# On the Method of Subspace Correction

Jinchao Xu

Center for Computational Mathematics and Applications
Department of Mathematics Pennsylvania State University
xu@math.psu.edu, http://www.math.psu.edu/xu/

**Acknowledgment:** NSF, NSFC and A. Humboldt Foundation

Juresalum January 2008

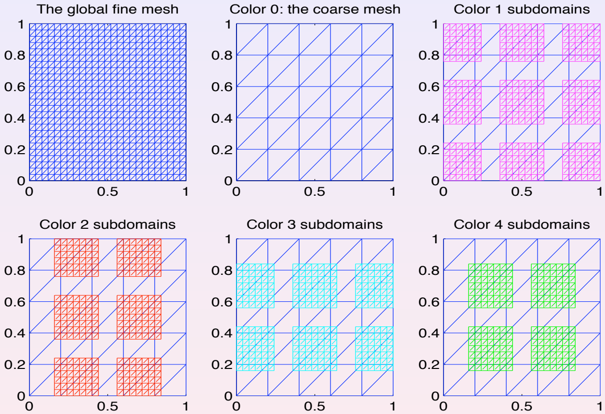
# Outline


1. Prologue: overlapping domain decomposition method
2. Convergence analysis for overlapping DDM
3. The method of subspace correction: framework and theory
4. Method of alternating projection
5. Interpretation of MG as the method of subspace correction
6. On problems with strongly discontinuous jumps
7. Auxiliary space method — the Method of Auxiliary-space correction
8. Epilogue

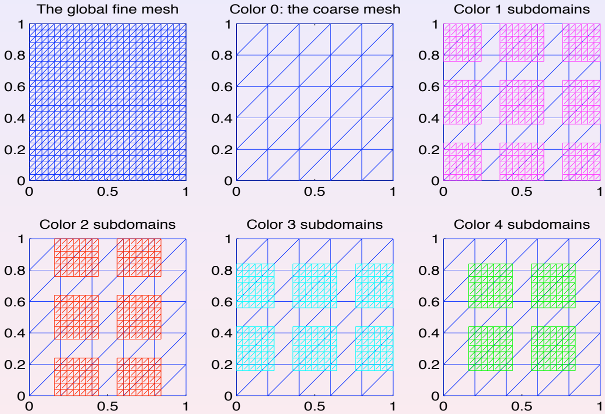
# Outline

1. Prologue: overlapping domain decomposition method
2. Convergence analysis for overlapping DDM
3. The method of subspace correction: framework and theory
4. Method of alternating projection
5. Interpretation of MG as the method of subspace correction
6. On problems with strongly discontinuous jumps
7. Auxiliary space method — the Method of Auxiliary-space correction
8. Epilogue

# Prologue: overlapping domain decomposition method



# Prologue: overlapping domain decomposition method




Define local subsapces:

$V_i = \{v \in V : v(x) = 0, \ \forall x \in \Omega \setminus \Omega_i\} \subset V \equiv H_0^1(\Omega) \ (1 \leq i \leq J)$

# Outline


1. Prologue: overlapping domain decomposition method
2. Convergence analysis for overlapping DDM
3. The method of subspace correction: framework and theory
4. Method of alternating projection
5. Interpretation of MG as the method of subspace correction
6. On problems with strongly discontinuous jumps
7. Auxiliary space method — the Method of Auxiliary-space correction
8. Epilogue

# Some earlier papers on the THEORY of overlapping DDM

1. Schwarz 1870
2. Dryja and Widlund 1987
3. Lions 1988 (related: von Neumann 1933)
4. Widlund 1988, T. Mathew (1989)
5. Bramble, Wang, Pasciak and Xu 1991

# Outline


1. Prologue: overlapping domain decomposition method
2. Convergence analysis for overlapping DDM
3. The method of subspace correction: framework and theory
4. Method of alternating projection
5. Interpretation of MG as the method of subspace correction
6. On problems with strongly discontinuous jumps
7. Auxiliary space method — the Method of Auxiliary-space correction
8. Epilogue

# General iterative methods

$$Au = f$$

An iterative method can be obtained through 3 steps

# General iterative methods

$$Au = f$$

An iterative method can be obtained through 3 steps

1. Form the residual: $r = f - Au^{k-1}$

# General iterative methods

$$Au = f$$

An iterative method can be obtained through 3 steps

1. Form the residual: $r = f - Au^{k-1}$
2. Solve the residual equation $Ae = r$

# General iterative methods

$$Au = f$$

An iterative method can be obtained through 3 steps

1. Form the residual: $r = f - Au^{k-1}$
2. Solve the residual equation $Ae = r$ approximately $\hat{e} = Br$ with $B \approx A^{-1}$
3. Update $u^k = u^{k-1} + \hat{e}$

# General iterative methods

$$Au = f$$

An iterative method can be obtained through 3 steps

1. Form the residual: $r = f - Au^{k-1}$
2. Solve the residual equation $Ae = r$ approximately $\hat{e} = Br$ with $B \approx A^{-1}$
3. Update $u^k = u^{k-1} + \hat{e}$

Example. Assume $A = (a_{ij}) \in R^{n\times n}$ and $A = D - L - U$. We may take

$$B = D^{-1}(\text{Jacobi}) \;\; \& \;\; or B = (D-L)^{-1}(\text{Gauss-Seidel}).$$

# The method of subspace corrections (MSC)

(Xu, SIAM Review 1992)

# The method of subspace corrections (MSC)

(Xu, SIAM Review 1992)

- Find $u \in V$ such that $a(u, v) = f(v), \ \forall v \in V$.
  - (e.g. $Au = f \Leftrightarrow (Au, v) = (f, v), \ \forall v \in R^J$)

# The method of subspace corrections (MSC)

(Xu, SIAM Review 1992)

- Find $u \in V$ such that $a(u, v) = f(v), \ \forall v \in V$.
  - (e.g. $Au = f \Leftrightarrow (Au, v) = (f, v), \ \forall v \in R^J$)

# The method of subspace corrections (MSC)

(Xu, SIAM Review 1992)

- Find $u \in V$ such that $a(u, v) = f(v),\ \forall v \in V$.
  - (e.g. $Au = f \Leftrightarrow (Au, v) = (f, v),\ \forall v \in R^J$)
- Space decomposition: $V = \sum_i V_i$
  - (e.g. $R^J = \sum_{i=1}^J \{e_i\}$)

# The method of subspace corrections (MSC)

(Xu, SIAM Review 1992)

- Find $u \in V$ such that $a(u, v) = f(v), \ \forall v \in V$.
  - (e.g. $Au = f \Leftrightarrow (Au, v) = (f, v), \ \forall v \in R^J$)
- Space decomposition: $V = \sum_i V_i$
  - (e.g. $R^J = \sum_{i=1}^{J} \{e_i\}$)
- Successive subspace correction (multilicative Schwarz):
  $u \leftarrow u + e_i \quad \text{for } i = 1 : J$
  where $e_i \in V_i$ solves $a_i(e_i, v_i) = f(v_i) - a(u, v_i) \quad \forall v_i \in V_i$.
  - (e.g. Gauss-Seidel: $u \leftarrow u + (D - L)^{-1}(b - Au)$)

# The method of subspace corrections (MSC)

(Xu, SIAM Review 1992)

- Find $u \in V$ such that $a(u, v) = f(v), \ \forall v \in V$.
  - (e.g. $Au = f \Leftrightarrow (Au, v) = (f, v), \ \forall v \in R^J$)
- Space decomposition: $V = \sum_i V_i$
  - (e.g. $R^J = \sum_{i=1}^J \{e_i\}$)
- Successive subspace correction (multilicative Schwarz):
  $u \leftarrow u + e_i \quad$ for $i = 1 : J$
  where $e_i \in V_i$ solves $a_i(e_i, v_i) = f(v_i) - a(u, v_i) \quad \forall v_i \in V_i$.
  - (e.g. Gauss-Seidel: $u \leftarrow u + (D - L)^{-1}(b - Au)$)
- Parallel subspace correction (additive Schwarz, BPX preconditioner):

$$u \leftarrow u + B(f - Au), \qquad B = \sum_{i=1}^J I_i R_i I_i^T = \sum_{i=1}^J R_i Q_i.$$

  - (e.g. Jacobi: $u \leftarrow u + D^{-1}(b - Au)$)

# Examples of MSC

# Examples of MSC

# Examples of MSC

1. Jacobi and Gauss-Seidel methods
    - interpret simple algorithms in a complicated way!

# Examples of MSC

1. Jacobi and Gauss-Seidel methods
   - interpret simple algorithms in a complicated way!
2. Overlapping domain decomposition methods
   - mostly horizontal decomposition with a little bit overlapping: inexact (but good) subspace solvers
   - (historically) inspiring example

# Examples of MSC

1. Jacobi and Gauss-Seidel methods
   - interpret simple algorithms in a complicated way!
2. Overlapping domain decomposition methods
   - mostly horizontal decomposition with a little bit overlapping: inexact (but good) subspace solvers
   - (historically) inspiring example
3. Nonoverallping domain decomposition methods
   - mostly horizontal decomposition with no overlapping: exact subspace solvers
   - one key (nontrivial) issue: construction of coarse spaces

# Examples of MSC

1. Jacobi and Gauss-Seidel methods
   - interpret simple algorithms in a complicated way!
2. Overlapping domain decomposition methods
   - mostly horizontal decomposition with a little bit overlapping: inexact (but good) subspace solvers
   - (historically) inspiring example
3. Nonoverallping domain decomposition methods
   - mostly horizontal decomposition with no overlapping: exact subspace solvers
   - one key (nontrivial) issue: construction of coarse spaces
4. Multigrid methods
   - mostly vertical decomposition with a lot of overlappings: very simple subspace solvers (smoothers)
   - nontrivial to be viewed as MSC
   - interpretation as MSC is one key development in multigrid theory

# Examples of MSC

1. Jacobi and Gauss-Seidel methods
   - interpret simple algorithms in a complicated way!
2. Overlapping domain decomposition methods
   - mostly horizontal decomposition with a little bit overlapping: inexact (but good) subspace solvers
   - (historically) inspiring example
3. Nonoverallping domain decomposition methods
   - mostly horizontal decomposition with no overlapping: exact subspace solvers
   - one key (nontrivial) issue: construction of coarse spaces
4. Multigrid methods
   - mostly vertical decomposition with a lot of overlappings: very simple subspace solvers (smoothers)
   - nontrivial to be viewed as MSC
   - interpretation as MSC is one key development in multigrid theory
5. Method of alternating projections

# Examples of MSC

1. Jacobi and Gauss-Seidel methods
   - interpret simple algorithms in a complicated way!
2. Overlapping domain decomposition methods
   - mostly horizontal decomposition with a little bit overlapping: inexact (but good) subspace solvers
   - (historically) inspiring example
3. Nonoverallping domain decomposition methods
   - mostly horizontal decomposition with no overlapping: exact subspace solvers
   - one key (nontrivial) issue: construction of coarse spaces
4. Multigrid methods
   - mostly vertical decomposition with a lot of overlappings: very simple subspace solvers (smoothers)
   - nontrivial to be viewed as MSC
   - interpretation as MSC is one key development in multigrid theory
5. Method of alternating projections
6. Other examples
   - Most existing or newly developed algorithms can be viewed as MSC in some sense (including nonlinear problems, e.g. Korhumber (obstacle problems))

# Some examples of convergence analysis

- Gauss-Seidel type methods
- Domain decomposition
  - Original Schwarz alternating iteration (1870)
  - P.L. Lions (1987) ($J = 2$)
  - Dryja and Widlund (1991) (additive)
  - Bramble, Pasciak, Wang and Xu (1991)
- Multigrid
  - Fedorenko (1961), Bachvalov (1966)
  - Brandt (1976, 1977), Hackbusch (1976), Bank-Dupont (1977), Nicolaides(1977)
  - Braess and Hackbusch (1983)
  - Bramble, Pasciak, Wang and Xu (1991)
- General framework of MSC and theory
  - Xu (1992), SIAM Review
  - Xu and Zikatanov (2002), J. of AMS
  - Xu and Zikatanov (2005)
  - Lee, Wu, Xu and Zikatanov (2005), Math. Comp.
- The method of alternating projections ...

# Outline


1. Prologue: overlapping domain decomposition method
2. Convergence analysis for overlapping DDM
3. The method of subspace correction: framework and theory
4. Method of alternating projection
5. Interpretation of MG as the method of subspace correction
6. On problems with strongly discontinuous jumps
7. Auxiliary space method — the Method of Auxiliary-space correction
8. Epilogue

# Alternating projection method

Let $H$ be a Hilbert space and $M_i \subset H$ be closed subspaces

- On the product of two projections: $P_{M_2} P_{M_1}$ is a projection if and only if $P_{M_2} P_{M_1} = P_{M_1} P_{M_2}$, furthermore $P_{M_2} P_{M_1} = P_{M_1 \cap M_2}$.
- von Neumann (1933):

# Alternating projection method

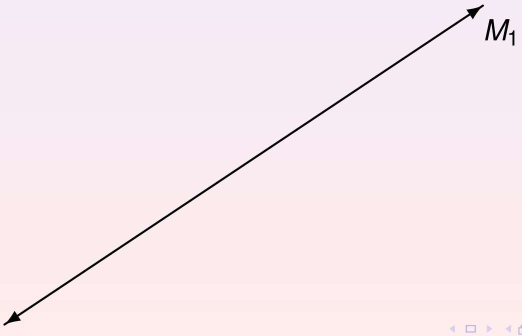
Let $H$ be a Hilbert space and $M_i \subset H$ be closed subspaces

- On the product of two projections: $P_{M_2}P_{M_1}$ is a projection if and only if $P_{M_2}P_{M_1} = P_{M_1}P_{M_2}$, furthermore $P_{M_2}P_{M_1} = P_{M_1 \cap M_2}$.
- von Neumann (1933): $\lim_{k\to\infty}(P_{M_2}P_{M_1})^k = P_{M_1 \cap M_2}$.

# Alternating projection method

Let $H$ be a Hilbert space and $M_i \subset H$ be closed subspaces

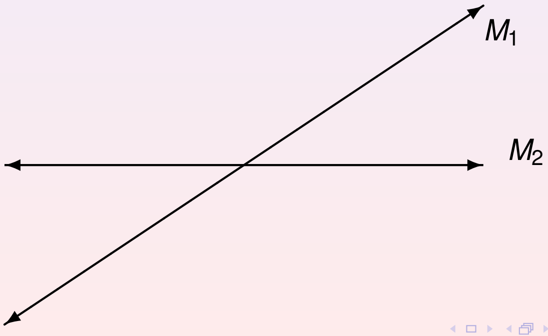
- On the product of two projections: $P_{M_2}P_{M_1}$ is a projection if and only if $P_{M_2}P_{M_1} = P_{M_1}P_{M_2}$, furthermore $P_{M_2}P_{M_1} = P_{M_1 \cap M_2}$.
- von Neumann (1933): $\lim_{k\to\infty}(P_{M_2}P_{M_1})^k = P_{M_1\cap M_2}$.



# Alternating projection method

Let $H$ be a Hilbert space and $M_i \subset H$ be closed subspaces

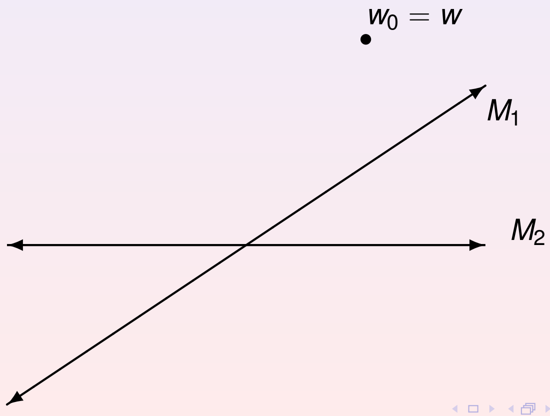
- On the product of two projections: $P_{M_2}P_{M_1}$ is a projection if and only if $P_{M_2}P_{M_1} = P_{M_1}P_{M_2}$, furthermore $P_{M_2}P_{M_1} = P_{M_1 \cap M_2}$.
- von Neumann (1933): $\lim_{k\to\infty}(P_{M_2}P_{M_1})^k = P_{M_1 \cap M_2}$.



# Alternating projection method

Let $H$ be a Hilbert space and $M_i \subset H$ be closed subspaces

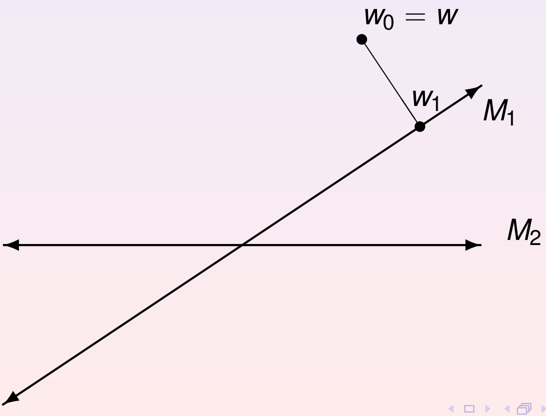
- On the product of two projections: $P_{M_2}P_{M_1}$ is a projection if and only if $P_{M_2}P_{M_1} = P_{M_1}P_{M_2}$, furthermore $P_{M_2}P_{M_1} = P_{M_1 \cap M_2}$.
- von Neumann (1933): $\lim_{k\to\infty}(P_{M_2}P_{M_1})^k = P_{M_1 \cap M_2}$.



# Alternating projection method

Let $H$ be a Hilbert space and $M_i \subset H$ be closed subspaces

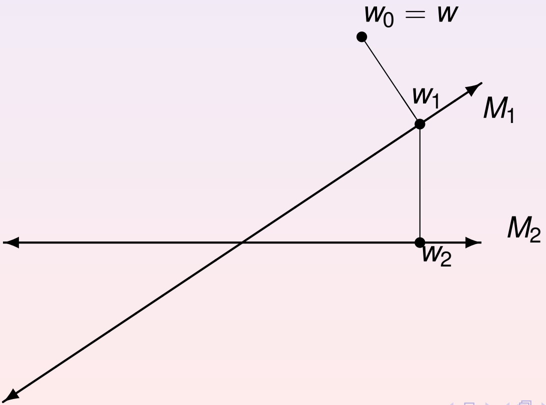
- On the product of two projections: $P_{M_2}P_{M_1}$ is a projection if and only if $P_{M_2}P_{M_1} = P_{M_1}P_{M_2}$, furthermore $P_{M_2}P_{M_1} = P_{M_1 \cap M_2}$.
- von Neumann (1933): $\lim_{k\to\infty}(P_{M_2}P_{M_1})^k = P_{M_1\cap M_2}$.



# Alternating projection method

Let $H$ be a Hilbert space and $M_i \subset H$ be closed subspaces

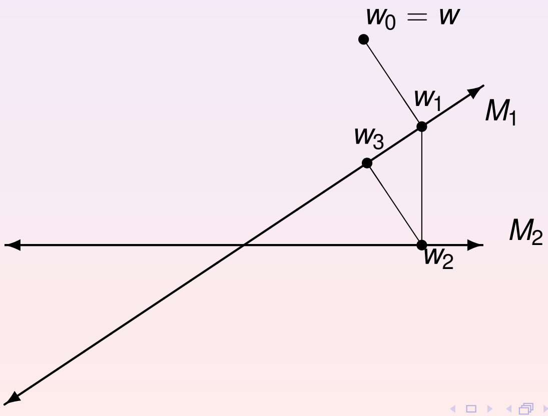
- On the product of two projections: $P_{M_2}P_{M_1}$ is a projection if and only if $P_{M_2}P_{M_1} = P_{M_1}P_{M_2}$, furthermore $P_{M_2}P_{M_1} = P_{M_1\cap M_2}$.
- von Neumann (1933): $\lim_{k\to\infty}(P_{M_2}P_{M_1})^k = P_{M_1\cap M_2}$.



# Alternating projection method

Let $H$ be a Hilbert space and $M_i \subset H$ be closed subspaces

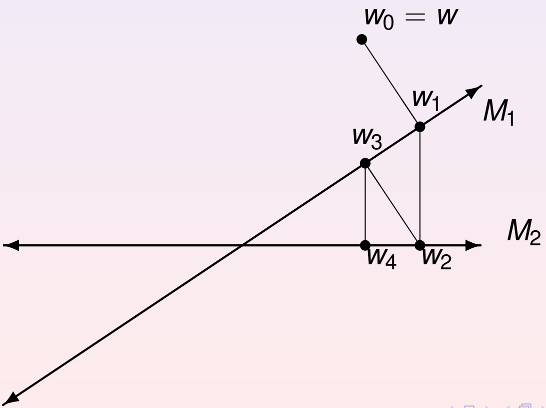
- On the product of two projections: $P_{M_2}P_{M_1}$ is a projection if and only if $P_{M_2}P_{M_1} = P_{M_1}P_{M_2}$, furthermore $P_{M_2}P_{M_1} = P_{M_1 \cap M_2}$.
- von Neumann (1933): $\lim_{k\to\infty}(P_{M_2}P_{M_1})^k = P_{M_1\cap M_2}$.



# Alternating projection method

Let $H$ be a Hilbert space and $M_i \subset H$ be closed subspaces

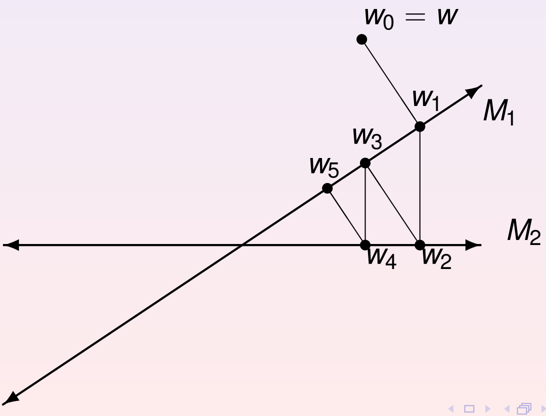
- On the product of two projections: $P_{M_2} P_{M_1}$ is a projection if and only if $P_{M_2} P_{M_1} = P_{M_1} P_{M_2}$, furthermore $P_{M_2} P_{M_1} = P_{M_1 \cap M_2}$.
- von Neumann (1933): $\lim_{k\to\infty} (P_{M_2} P_{M_1})^k = P_{M_1 \cap M_2}$.



# Alternating projection method

Let $H$ be a Hilbert space and $M_i \subset H$ be closed subspaces

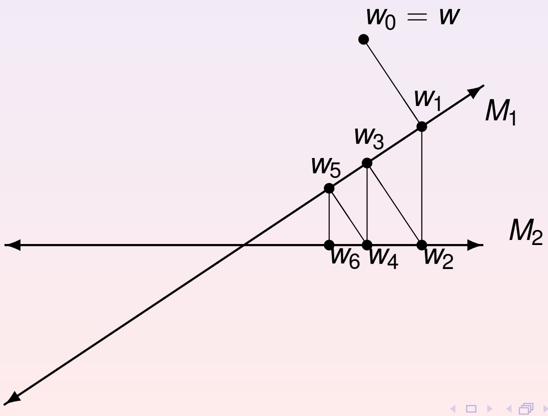
- On the product of two projections: $P_{M_2}P_{M_1}$ is a projection if and only if $P_{M_2}P_{M_1} = P_{M_1}P_{M_2}$, furthermore $P_{M_2}P_{M_1} = P_{M_1\cap M_2}$.
- von Neumann (1933): $\lim_{k\to\infty}(P_{M_2}P_{M_1})^k = P_{M_1\cap M_2}$.



# Alternating projection method

Let $H$ be a Hilbert space and $M_i \subset H$ be closed subspaces

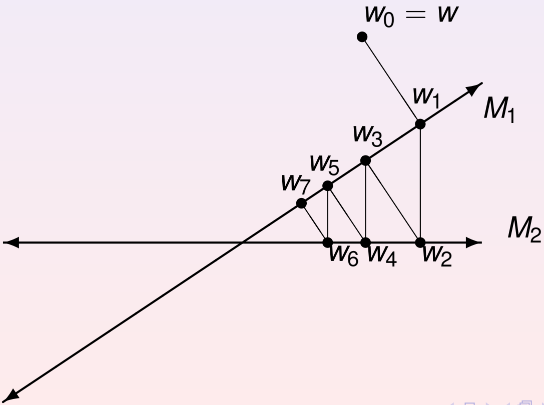
- On the product of two projections: $P_{M_2}P_{M_1}$ is a projection if and only if $P_{M_2}P_{M_1} = P_{M_1}P_{M_2}$, furthermore $P_{M_2}P_{M_1} = P_{M_1 \cap M_2}$.
- von Neumann (1933): $\lim_{k\to\infty}(P_{M_2}P_{M_1})^k = P_{M_1 \cap M_2}$.



# Alternating projection method

Let $H$ be a Hilbert space and $M_i \subset H$ be closed subspaces

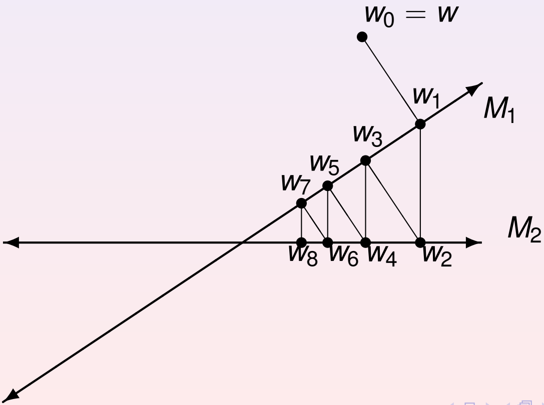
- On the product of two projections: $P_{M_2} P_{M_1}$ is a projection if and only if $P_{M_2} P_{M_1} = P_{M_1} P_{M_2}$, furthermore $P_{M_2} P_{M_1} = P_{M_1 \cap M_2}$.
- von Neumann (1933): $\lim_{k \to \infty} (P_{M_2} P_{M_1})^k = P_{M_1 \cap M_2}$.



# Alternating projection method

Let $H$ be a Hilbert space and $M_i \subset H$ be closed subspaces

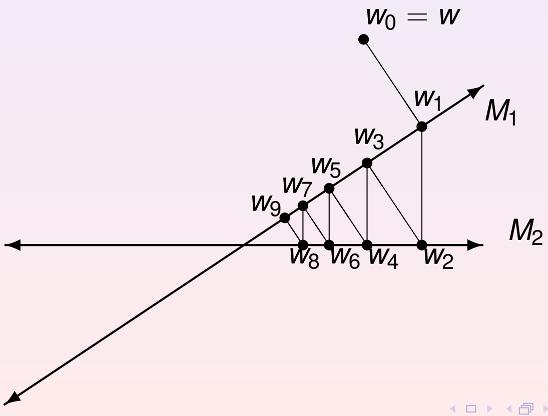
- On the product of two projections: $P_{M_2}P_{M_1}$ is a projection if and only if $P_{M_2}P_{M_1} = P_{M_1}P_{M_2}$, furthermore $P_{M_2}P_{M_1} = P_{M_1 \cap M_2}$.
- von Neumann (1933): $\lim_{k\to\infty}(P_{M_2}P_{M_1})^k = P_{M_1 \cap M_2}$.



# Alternating projection method

Let $H$ be a Hilbert space and $M_i \subset H$ be closed subspaces

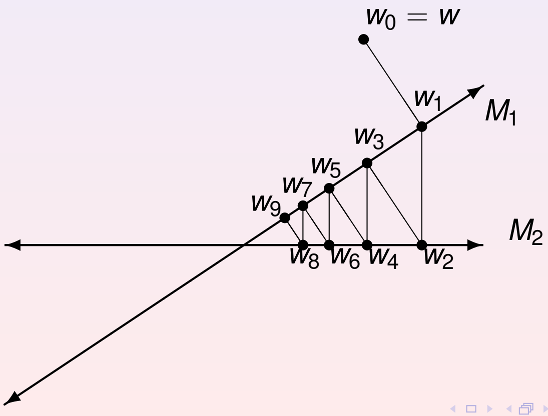
- On the product of two projections: $P_{M_2}P_{M_1}$ is a projection if and only if $P_{M_2}P_{M_1} = P_{M_1}P_{M_2}$, furthermore $P_{M_2}P_{M_1} = P_{M_1 \cap M_2}$.
- von Neumann (1933): $\lim_{k\to\infty}(P_{M_2}P_{M_1})^k = P_{M_1\cap M_2}$.



# Alternating projection method

Let $H$ be a Hilbert space and $M_i \subset H$ be closed subspaces

- On the product of two projections: $P_{M_2} P_{M_1}$ is a projection if and only if $P_{M_2} P_{M_1} = P_{M_1} P_{M_2}$, furthermore $P_{M_2} P_{M_1} = P_{M_1 \cap M_2}$.
- von Neumann (1933): $\lim_{k\to\infty} (P_{M_2} P_{M_1})^k = P_{M_1 \cap M_2}$.



- Rate of convergence:

- Rate of convergence:

$$\|(P_{M_2}P_{M_1})^k - P_{M_1\cap M_2}\| = [c(M_1, M_2)]^{2k-1}$$

- Rate of convergence:

$$\|(P_{M_2}P_{M_1})^k - P_{M_1\cap M_2}\| = [c(M_1, M_2)]^{2k-1}$$

where $c(M_1, M_2)$ is the cosine of the angle between $M_1$ and $M_2$:

$$c(M_1, M_2) = \sup\left\{\frac{|(u, v)|}{\|u\|\|v\|} : u \in M_1 \cap (M_1 \cap M_2)^{\perp}, v \in M_2 \cap (M_1 \cap M_2$$

- Rate of convergence:

$$\|(P_{M_2}P_{M_1})^k - P_{M_1\cap M_2}\| = [c(M_1, M_2)]^{2k-1}$$

where $c(M_1, M_2)$ is the cosine of the angle between $M_1$ and $M_2$:

$$c(M_1, M_2) = \sup\left\{\frac{|(u, v)|}{\|u\|\|v\|} : u \in M_1 \cap (M_1 \cap M_2)^{\perp}, v \in M_2 \cap (M_1 \cap M_2)^{\perp}\right\}$$

Fact: $c(M_1, M_2) < 1 \Leftrightarrow$ if $M_1 + M_2$ is closed. $\Leftrightarrow$ uniform convergence.

# Generalization to more than two subspaces

- Given any closed subspaces $M_i$,
  $\lim_{k\to\infty} w_k = \lim_{k\to\infty} (\prod_{i=1}^{J} P_{M_i})^k w = P_{\cap_{i=1}^{J} M_i} w.$
- If $\sum_i M_i$ is close, then

$$\lim_{k\to\infty} \prod_{i=1}^{J} P_{M_i})^k = P_{\cap_{i=1}^{J} M_i}.$$

- Convergence estimate: in terms of “pairwise angles”?

# Works related to MAP

- Two subspaces:
  - von Neumann (1933)
  - N. Aronszjan (1950)
  - I. Halperin (1962)
- More general results (subspaces $> 2$, convex sets, etc):
  - F. Deutsch (1982, 1983, 1985, 1992)
  - S. Kayalar and H. Weinert (1988)
  - H. Bauschke and J. Borwein (1996) (SIAM Review)
- Most of the existing convergence estimates are in terms of angles between every pair of subspaces.

# The relationship between MAP and MSC

- An auxiliary result: Let $M_i \subset H$ and $M = \cap_{i=1}^{J} M_i$. Then

$$M^\perp = \sum_{i=1}^{J} M_i^\perp$$

# The relationship between MAP and MSC

- An auxiliary result: Let $M_i \subset H$ and $M = \cap_{i=1}^{J} M_i$. Then
  $$M^\perp = \sum_{i=1}^{J} M_i^\perp$$
- An equivalence result (Xu and Zikatanov 2000)
  - Algorithm MAP is equivalent to subspace correction method with $a(\cdot,\cdot) = (\cdot,\cdot)_H$, $T_i = P_i$ and $u^0 = 0$, if $w$ and $M_i$ in Algorithm MAP and $f$ and $V_i$ in Algorithm MSC are related by $(w,\phi)_H = f(\phi) \quad \forall \phi \in H \quad \text{and} \quad V_i = M_i^\perp$. One also has that: $u = P_V w = P_M^\perp w, u^\ell = w - w^\ell$.

# Convergence estimates for MSC

1. Parallel subspace corrections

$$(B^{-1}v, v) = \inf_{\sum_i v_i = v} \sum_i (R_i^{-1} v_i, v_i)$$

# Convergence estimates for MSC

1. Parallel subspace corrections

$$(B^{-1}v, v) = \inf_{\sum_i v_i = v} \sum_i (R_i^{-1} v_i, v_i)$$

or

$$((BA)^{-1}v, v)_A = \inf_{\sum_i v_i = v} \sum_i ((R_i A_i)^{-1} v_i, v_i)_A$$

# Convergence estimates for MSC

1. Parallel subspace corrections

$$(B^{-1}v, v) = \inf_{\sum_i v_i = v} \sum_i (R_i^{-1} v_i, v_i)$$

or

$$((BA)^{-1}v, v)_A = \inf_{\sum_i v_i = v} \sum_i ((R_i A_i)^{-1} v_i, v_i)_A$$

2. Successive subspace corrections (SSC)
   - Error transfer operator:

$$u - u^\ell = E(u - u^{\ell-1}) = \ldots = E^\ell (u - u^0)$$

where $E = (I - T_J)(I - T_{J-1}) \ldots (I - T_1)$.

# Convergence estimates for MSC

1. Parallel subspace corrections

$$(B^{-1}v, v) = \inf_{\sum_i v_i = v} \sum_i (R_i^{-1} v_i, v_i)$$

or

$$((BA)^{-1}v, v)_A = \inf_{\sum_i v_i = v} \sum_i ((R_i A_i)^{-1} v_i, v_i)_A$$

2. Successive subspace corrections (SSC)
   - Error transfer operator:

$$u - u^\ell = E(u - u^{\ell-1}) = \ldots = E^\ell (u - u^0)$$

     where $E = (I - T_J)(I - T_{J-1}) \ldots (I - T_1)$.
   - Convergence: $\|E\| < 1$?
   - Many estimates existed (BPWX-1991, X-1992, ...)

# A sharp convergence result

**Theorem** (XU AND ZIKATANOV 2002, J. AMS, 2005, 2007)

# A sharp convergence result

**Theorem** (XU AND ZIKATANOV 2002, J. AMS, 2005, 2007)

$$\|E\|^2_{\mathcal{L}(V,V)} \equiv \|(I - T_J)(I - T_{J-1}) \ldots (I - T_1)\|^2$$

# A sharp convergence result

**Theorem** (XU AND ZIKATANOV 2002, J. AMS, 2005, 2007)

$$\|E\|^2_{\mathcal{L}(V,V)} \equiv \|(I - T_J)(I - T_{J-1})\dots(I - T_1)\|^2 = \frac{c_0}{1 + c_0} = 1 - \frac{1}{K}$$

# A sharp convergence result

**Theorem** (XU AND ZIKATANOV 2002, J. AMS, 2005, 2007)

$$\|E\|^2_{\mathcal{L}(V,V)} \equiv \|(I-T_J)(I-T_{J-1})\dots(I-T_1)\|^2 = \frac{c_0}{1+c_0} = 1 - \frac{1}{K}$$

where $\bar{T}_i = T_i^* + T_i - T_i^* T_i$ and, with $w_i = \sum_{j=i}^J v_j - T_i^{-1} v_i$

$$c_0 = \sup_{\|v\|=1} \inf_{\sum_i v_i = v} \sum_{i=1}^J (T_i \bar{T}_i^{-1} T_i^* w_i, w_i)$$

and, with $v^i = \sum_{j=i+1}^J v_j$

$$K = 1 + c_0 = \sup_{\|v\|=1} \inf_{\sum_i v_i = v} \sum_{i=1}^J (\bar{T}_i^{-1}(v_i + T_i^* v^i), v_i + T_i^* v^i)$$

# A sharp convergence result

**Theorem** (XU AND ZIKATANOV 2002, J. AMS, 2005, 2007)

$$\|E\|^2_{\mathcal{L}(V,V)} \equiv \|(I - T_J)(I - T_{J-1})\dots(I - T_1)\|^2 = \frac{c_0}{1 + c_0} = 1 - \frac{1}{K}$$

where $\bar{T}_i = T_i^* + T_i - T_i^* T_i$ and, with $w_i = \sum_{j=i}^{J} v_j - T_i^{-1} v_i$

$$c_0 = \sup_{\|v\|=1} \inf_{\sum_i v_i = v} \sum_{i=1}^{J} (T_i \bar{T}_i^{-1} T_i^* w_i, w_i)$$

and, with $v^i = \sum_{j=i+1}^{J} v_j$

$$K = 1 + c_0 = \sup_{\|v\|=1} \inf_{\sum_i v_i = v} \sum_{i=1}^{J} (\bar{T}_i^{-1}(v_i + T_i^* v^i), v_i + T_i^* v^i)$$

$\bar{T}_i$ is SPD on $V_i$

# A sharp convergence result

**Theorem** (Xu and Zikatanov 2002, J. AMS, 2005, 2007)

$$\|E\|^2_{\mathcal{L}(V,V)} \equiv \|(I - T_J)(I - T_{J-1})\dots(I - T_1)\|^2 = \frac{c_0}{1 + c_0} = 1 - \frac{1}{K}$$

where $\bar{T}_i = T_i^* + T_i - T_i^* T_i$ and, with $w_i = \sum_{j=i}^J v_j - T_i^{-1} v_i$

$$c_0 = \sup_{\|v\|=1} \inf_{\sum_i v_i = v} \sum_{i=1}^J (T_i \bar{T}_i^{-1} T_i^* w_i, w_i)$$

and, with $v^i = \sum_{j=i+1}^J v_j$

$$K = 1 + c_0 = \sup_{\|v\|=1} \inf_{\sum_i v_i = v} \sum_{i=1}^J (\bar{T}_i^{-1}(v_i + T_i^* v^i), v_i + T_i^* v^i)$$

$\bar{T}_i$ is SPD on $V_i \Rightarrow c_0 > 0$

# A sharp convergence result

**Theorem** (Xu and Zikatanov 2002, J. AMS, 2005, 2007)

$$\|E\|^2_{\mathcal{L}(V,V)} \equiv \|(I - T_J)(I - T_{J-1}) \dots (I - T_1)\|^2 = \frac{c_0}{1 + c_0} = 1 - \frac{1}{K}$$

where $\bar{T}_i = T_i^* + T_i - T_i^* T_i$ and, with $w_i = \sum_{j=i}^{J} v_j - T_i^{-1} v_i$

$$c_0 = \sup_{\|v\|=1} \inf_{\sum_i v_i = v} \sum_{i=1}^{J} (T_i \bar{T}_i^{-1} T_i^* w_i, w_i)$$

and, with $v^i = \sum_{j=i+1}^{J} v_j$

$$K = 1 + c_0 = \sup_{\|v\|=1} \inf_{\sum_i v_i = v} \sum_{i=1}^{J} (\bar{T}_i^{-1}(v_i + T_i^* v^i), v_i + T_i^* v^i)$$

$\bar{T}_i$ is SPD on $V_i \Rightarrow c_0 > 0 \Rightarrow \|E\|_{\mathcal{L}(V,V)} < 1 \Rightarrow$ convergence.

# A sharp convergence result

**Theorem** (Xu and Zikatanov 2002, J. AMS, 2005, 2007)

$$\|E\|^2_{\mathcal{L}(V,V)} \equiv \|(I-T_J)(I-T_{J-1})\ldots(I-T_1)\|^2 = \frac{c_0}{1+c_0} = 1 - \frac{1}{K}$$

where $\bar{T}_i = T_i^* + T_i - T_i^* T_i$ and, with $w_i = \sum_{j=i}^J v_j - T_i^{-1} v_i$

$$c_0 = \sup_{\|v\|=1} \inf_{\sum_i v_i = v} \sum_{i=1}^J (T_i \bar{T}_i^{-1} T_i^* w_i, w_i)$$

and, with $v^i = \sum_{j=i+1}^J v_j$

$$K = 1 + c_0 = \sup_{\|v\|=1} \inf_{\sum_i v_i = v} \sum_{i=1}^J (\bar{T}_i^{-1}(v_i + T_i^* v^i), v_i + T_i^* v^i)$$

$\bar{T}_i$ is SPD on $V_i \Rightarrow c_0 > 0 \Rightarrow \|E\|_{\mathcal{L}(V,V)} < 1 \Rightarrow$ convergence.

Special case $T_i = P_i$: $\quad K = \sup_{\|v\|=1} \inf_{\sum_i v_i = v} \sum_{i=1}^J \|P_i \sum_{j=i}^J v_j\|^2.$

# A concise analysis of Schwarz DD method

for $-\mathrm{div}\,(a(x)\mathrm{grad}\;u) = f$

# A concise analysis of Schwarz DD method

for $-\mathrm{div}\,(a(x)\mathrm{grad}\,u) = f$

1. Partition of unity: $\sum_{i=1}^{J} \theta_i(x) \equiv 1, \quad \mathrm{supp}\theta_i \subset \Omega_i \cup \partial\Omega, \quad 0 \le \theta_i \le 1, \quad \max_{x\in\bar{\Omega}_i} |\nabla\theta_i(x)| \le c_1 h_0^{-1}$.

# A concise analysis of Schwarz DD method
for $-\mathrm{div}\,(a(x)\mathrm{grad}\,u) = f$

1. Partition of unity: $\sum_{i=1}^{J} \theta_i(x) \equiv 1, \quad \mathrm{supp}\theta_i \subset \Omega_i \cup \partial\Omega, \quad 0 \le \theta_i \le 1, \quad \max_{x\in\bar{\Omega}_i} |\nabla\theta_i(x)| \le c_1 h_0^{-1}$.
2. $L^2$ projection $Q_0 : V \mapsto V_0$: $h_0^{-1}\|v - Q_0 v\|_{0,\Omega} + |v - Q_0 v|_{1,\Omega} \le c_2 |v|_{1,\Omega}$.

# A concise analysis of Schwarz DD method

for $-\mathrm{div}\,(a(x)\mathrm{grad}\,u) = f$

1. Partition of unity: $\sum_{i=1}^{J} \theta_i(x) \equiv 1, \quad \mathrm{supp}\theta_i \subset \Omega_i \cup \partial\Omega, \quad 0 \le \theta_i \le 1, \quad \max_{x\in\bar{\Omega}_i} |\nabla\theta_i(x)| \le c_1 h_0^{-1}$.
2. $L^2$ projection $Q_0 : V \mapsto V_0$: $h_0^{-1}\|v - Q_0 v\|_{0,\Omega} + |v - Q_0 v|_{1,\Omega} \le c_2 |v|_{1,\Omega}$.
3. $c_0 \le \frac{\Lambda_1}{\Lambda_0} C_0$ with $C_0 = C_0(c_1, c_2)$ independent of $a_{ij}$ and $J$:

$$\sum_{k=0}^{J} \|P_k \sum_{i=k+1}^{J} v_i\|_{a,\Omega}^2 \le \|v - Q_0 v\|_{a,\Omega}^2 + \sum_{k=1}^{J} \|(\sum_{i=k+1}^{J} \theta_i)(v - Q_0 v))\|_{a,\Omega_k}^2$$

$$\le \Lambda_1 \left( |v - Q_0 v|_{1,\Omega}^2 + \sum_{k=1}^{J} \max_{x\in\bar{\Omega}_k} |\sum_{i=k+1}^{J} \nabla\theta_i(x)| \|v - Q_0 v\|_{0,\Omega_k}^2 + |v - Q_0 v|_{1,\Omega_k}^2 \right)$$

$$\le \Lambda_1 C_0 |v|_{1,\Omega}^2 \le \frac{\Lambda_1}{\Lambda_0} C_0 \|v\|_{a,\Omega}^2$$

# A concise analysis of Schwarz DD method

for $-\mathrm{div}\,(a(x)\mathrm{grad}\,u) = f$

1. Partition of unity: $\sum_{i=1}^{J} \theta_i(x) \equiv 1, \quad \mathrm{supp}\theta_i \subset \Omega_i \cup \partial\Omega, \quad 0 \le \theta_i \le 1, \quad \max_{x\in\bar{\Omega}_i} |\nabla\theta_i(x)| \le c_1 h_0^{-1}$.
2. $L^2$ projection $Q_0 : V \mapsto V_0$: $h_0^{-1}\|v - Q_0 v\|_{0,\Omega} + |v - Q_0 v|_{1,\Omega} \le c_2 |v|_{1,\Omega}$.
3. $c_0 \le \frac{\Lambda_1}{\Lambda_0} C_0$ with $C_0 = C_0(c_1, c_2)$ independent of $a_{ij}$ and $J$:

$$\sum_{k=0}^{J} \|P_k \sum_{i=k+1}^{J} v_i\|_{a,\Omega}^2 \le \|v - Q_0 v\|_{a,\Omega}^2 + \sum_{k=1}^{J} \|(\sum_{i=k+1}^{J} \theta_i)(v - Q_0 v))\|_{a,\Omega_k}^2$$

$$\le \Lambda_1 \left( |v - Q_0 v|_{1,\Omega}^2 + \sum_{k=1}^{J} \max_{x\in\bar{\Omega}_k} |\sum_{i=k+1}^{J} \nabla\theta_i(x)| \|v - Q_0 v\|_{0,\Omega_k}^2 + |v - Q_0 v|_{1,\Omega_k}^2 \right)$$

$$\le \Lambda_1 C_0 |v|_{1,\Omega}^2 \le \frac{\Lambda_1}{\Lambda_0} C_0 \|v\|_{a,\Omega}^2$$

**Note:** *convergence rate does not depend on the smoothness or possible oscillations in $a_{ij}$! this is also true for multigrid convergence.*

# Outline

1 Prologue: overlapping domain decomposition method

2 Convergence analysis for overlapping DDM

3 The method of subspace correction: framework and theory

4 Method of alternating projection

5 Interpretation of MG as the method of subspace correction

6 On problems with strongly discontinuous jumps

7 Auxiliary space method — the Method of Auxiliary-space correction

8 Epilogue

# Interpretation of MG as MSC

My first attempted project (1986) on multigrid analysis:

> *For* $-\nabla \cdot (a(x)\nabla u) = f$ *with discontinuous coefficient a (in two dimensions), prove (or disprove) that the multigrid converges uniformly with respect to both mesh parameters and jump size (in a)*

# Interpretation of MG as MSC

My first attempted project (1986) on multigrid analysis:

> *For* $-\nabla \cdot (a(x)\nabla u) = f$ *with discontinuous coefficient a (in two dimensions), prove (or disprove) that the multigrid converges uniformly with respect to both mesh parameters and jump size (in a)*
>
> *(Final completion: 2007!)*

Observations and facts:

# Interpretation of MG as MSC

My first attempted project (1986) on multigrid analysis:

*For $-\nabla \cdot (a(x)\nabla u) = f$ with discontinuous coefficient a (in two dimensions), prove (or disprove) that the multigrid converges uniformly with respect to both mesh parameters and jump size (in a)*

*(Final completion: 2007!)*

Observations and facts:

1. Numerical examples (2D) indicate the convergence is indeed uniform

# Interpretation of MG as MSC

My first attempted project (1986) on multigrid analysis:

*For $-\nabla \cdot (a(x)\nabla u) = f$ with discontinuous coefficient a (in two dimensions), prove (or disprove) that the multigrid converges uniformly with respect to both mesh parameters and jump size (in a)*

*(Final completion: 2007!)*

Observations and facts:

1. Numerical examples (2D) indicate the convergence is indeed uniform
2. Earlier convergence techniques all depend on elliptic regularity, which can not be applied here because of lack of regularity (uniform with respect to jumps)

# Interpretation of MG as MSC

My first attempted project (1986) on multigrid analysis:

*For $-\nabla \cdot (a(x)\nabla u) = f$ with discontinuous coefficient a (in two dimensions), prove (or disprove) that the multigrid converges uniformly with respect to both mesh parameters and jump size (in a)*

*(Final completion: 2007!)*

Observations and facts:

1. Numerical examples (2D) indicate the convergence is indeed uniform
2. Earlier convergence techniques all depend on elliptic regularity, which can not be applied here because of lack of regularity (uniform with respect to jumps)
3. A completely new theory is needed

# Interpretation of MG as MSC

My first attempted project (1986) on multigrid analysis:

*For $-\nabla \cdot (a(x)\nabla u) = f$ with discontinuous coefficient $a$ (in two dimensions), prove (or disprove) that the multigrid converges uniformly with respect to both mesh parameters and jump size (in $a$)*

*(Final completion: 2007!)*

Observations and facts:

1. Numerical examples (2D) indicate the convergence is indeed uniform
2. Earlier convergence techniques all depend on elliptic regularity, which can not be applied here because of lack of regularity (uniform with respect to jumps)
3. A completely new theory is needed
4. First key (historic) step: interpreted MG (originally always defined by recursion) as an MSC

# Regularity-free theory for multigrid methods

(Bramble, Pasiak, Wang and Xu 1991) Using the framework and theory of MSC, it is possible to analyze multigrid convergence with little or without any regularity. As a result, we can analyze:

# Regularity-free theory for multigrid methods

(Bramble, Pasiak, Wang and Xu 1991) Using the framework and theory of MSC, it is possible to analyze multigrid convergence with little or without any regularity. As a result, we can analyze:

# Regularity-free theory for multigrid methods

(Bramble, Pasiak, Wang and Xu 1991) Using the framework and theory of MSC, it is possible to analyze multigrid convergence with little or without any regularity. As a result, we can analyze:

- locally refined grids (including bisection refinements)

# Regularity-free theory for multigrid methods

(Bramble, Pasiak, Wang and Xu 1991) Using the framework and theory of MSC, it is possible to analyze multigrid convergence with little or without any regularity. As a result, we can analyze:

- locally refined grids (including bisection refinements)
- discontinuous coefficients

# Regularity-free theory for multigrid methods

(Bramble, Pasiak, Wang and Xu 1991) Using the framework and theory of MSC, it is possible to analyze multigrid convergence with little or without any regularity. As a result, we can analyze:

- locally refined grids (including bisection refinements)
- discontinuous coefficients

A multilevel norm equivalence theory ($v_\ell = (Q_\ell - Q_{\ell-1})v$)

$$\|v\|_{H^1(\Omega)}^2 \eqsim \sum_{\ell=1}^{\infty} \|v_\ell\|_{H^1(\Omega)}^2$$

# Regularity-free theory for multigrid methods

(Bramble, Pasiak, Wang and Xu 1991) Using the framework and theory of MSC, it is possible to analyze multigrid convergence with little or without any regularity. As a result, we can analyze:

- locally refined grids (including bisection refinements)
- discontinuous coefficients

A multilevel norm equivalence theory ($v_\ell = (Q_\ell - Q_{\ell-1})v$)

$$\|v\|^2_{H^1(\Omega)} \eqsim \sum_{\ell=1}^{\infty} \|v_\ell\|^2_{H^1(\Omega)} \eqsim \sum_{\ell=1}^{\infty} h_\ell^{-2} \|v_\ell\|^2_{L^2(\Omega)}$$

and

# Regularity-free theory for multigrid methods

(Bramble, Pasiak, Wang and Xu 1991) Using the framework and theory of MSC, it is possible to analyze multigrid convergence with little or without any regularity. As a result, we can analyze:

- locally refined grids (including bisection refinements)
- discontinuous coefficients

A multilevel norm equivalence theory ($v_\ell = (Q_\ell - Q_{\ell-1})v$)

$$\|v\|^2_{H^1(\Omega)} \eqsim \sum_{\ell=1}^{\infty} \|v_\ell\|^2_{H^1(\Omega)} \eqsim \sum_{\ell=1}^{\infty} h_\ell^{-2} \|v_\ell\|^2_{L^2(\Omega)}$$

and

$$(-\Delta)^s \eqsim \sum_{\ell=1}^{\infty} h_\ell^{-2s} (Q_\ell - Q_{\ell-1}).$$

# Outline


1. Prologue: overlapping domain decomposition method
2. Convergence analysis for overlapping DDM
3. The method of subspace correction: framework and theory
4. Method of alternating projection
5. Interpretation of MG as the method of subspace correction
6. On problems with strongly discontinuous jumps
7. Auxiliary space method — the Method of Auxiliary-space correction
8. Epilogue

# On problems with strongly discontinuous jumps

# On problems with strongly discontinuous jumps

1. Nonoverlapping DD methods are known to be very robust with discontinuous jumps with construction of “exotic” coarse spaces
   - Toselli and Widlund book (2005), Xu and Zou (1998)

# On problems with strongly discontinuous jumps

1. Nonoverlapping DD methods are known to be very robust with discontinuous jumps with construction of "exotic" coarse spaces
   - Toselli and Widlund book (2005), Xu and Zou (1998)
2. It has been a long time open problem (in 3D) weather multigrid methods are robust with respect to jumps
   - Xu and Zhu 2007: *Multigrid methods (and overlapping DD, without using special coarse spaces) may not be robust w.r.t. jumps as an iterative method by itself, but it is robust if it is used as a preconditioner together with PCG method!*

# On problems with strongly discontinuous jumps

1. Nonoverlapping DD methods are known to be very robust with discontinuous jumps with construction of "exotic" coarse spaces
   - Toselli and Widlund book (2005), Xu and Zou (1998)
2. It has been a long time open problem (in 3D) weather multigrid methods are robust with respect to jumps
   - Xu and Zhu 2007: *Multigrid methods (and overlapping DD, without using special coarse spaces) may not be robust w.r.t. jumps as an iterative method by itself, but it is robust if it is used as a preconditioner together with PCG method!*
3. A more general question: robust iterative methods for nearly singular problems (Lee, Wu, Xu and Zikatanov 2005-7, see my talk tomorrow)

# Condition number or "effective condition number"?

## Condition number or "effective condition number"?

## Condition number or "effective condition number"?

- V-cycle multigrid (and standard overlapping Schwarz-DD) gives a deteriorating condition number, but robust effective condition number:

$$\frac{\lambda_N}{\lambda_{m_0}}, \qquad \lambda_{\min} = \lambda_1 \le \lambda_2 \le \ldots \le \lambda_N = \lambda_{\max}$$

($m_0$ = number of jumps)

## Condition number or “effective condition number”?

- V-cycle multigrid (and standard overlapping Schwarz-DD) gives a deteriorating condition number, but robust effective condition number:

$$\frac{\lambda_N}{\lambda_{m_0}}, \qquad \lambda_{\min} = \lambda_1 \le \lambda_2 \le \ldots \le \lambda_N = \lambda_{\max}$$

  ($m_0 =$ number of jumps)
- When PCG is applied, condition number is not always a good measure.

  *For nonoverlapping DD methods, exotic coarse spaces may be much simplified to still have robust “effective condition number”*

(see my talk tomorrow)

# Outline


1. Prologue: overlapping domain decomposition method
2. Convergence analysis for overlapping DDM
3. The method of subspace correction: framework and theory
4. Method of alternating projection
5. Interpretation of MG as the method of subspace correction
6. On problems with strongly discontinuous jumps
7. Auxiliary space method — the Method of Auxiliary-space correction
8. Epilogue

# Auxiliary space method

(Xu 1996)

$$u \in V(\text{Hilbert space}): \quad a(u,v) = f(v) \quad \forall v \in V\ .$$

Consider the following product space

$$\bar{V} = V \times W_1 \times \cdots \times W_J\ , \tag{1}$$

with the inner product: $\bar{a}(\bar{v},\bar{v}) := s(v_0,v_0) + \sum_{j=1}^{J} \bar{a}_j(w_j,w_j)$.
Here $W_1,\ldots,W_J$, $J \in \mathbb{N}$ are auxiliary (Hilbert) spaces endowed with inner products $\bar{a}_j\cdot,\cdot)$, $j = 1,\ldots,J$.
With $\Pi_j : W_j \mapsto V$, we have the auxiliary space preconditioner:

$$B = S^{-1} + \sum_{j=1}^{J} \Pi_j \bar{A}_j^{-1} \Pi_j^*$$

The space $V$ itself presents as a component of $\bar{V}$ is equipped with an inner product $s(\cdot,\cdot)$ different from $a(\cdot,\cdot)$. The operator $S : V \mapsto V$ induced by $s(\cdot,\cdot)$ on $V$ is usually called the smoother.
The auxiliary preconditioner admits the following estimate:

$$\kappa(BA) \le c_0^2(c_s^2 + c_1^2 + \cdots + c_J^2) \ .$$

where $\|\Pi_j w_j\|_A \le c_j \bar{a}_j(w_j, w_j)^{\frac{1}{2}} \ , \quad w_j \in W_j \ ,$
$\|v\|_A \le c_s s(v,v)^{\frac{1}{2}} \quad \forall v \in V$ , and for $v \in V$, $\exists\ v_0 \in V$ and $w_j \in W_j$ s. t. $v = v_0 + \sum_{j=1}^J \Pi_j w_j$ and

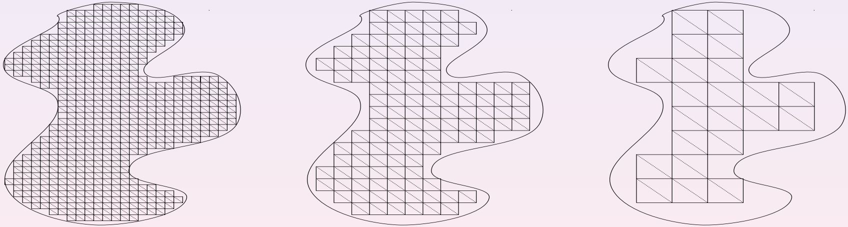
$$s(v_0, v_0) + \sum_{j=1}^{J} \bar{a}_j(w_j, w_j) \le c_0^2 \|v\|_A^2 \ ,$$

(related) fictitious domain (space) methods: see Glowinski's talk

# Example 1: “nearby” auxiliary space

(use structured grid to precondition unstructured grid)

- J. Xu, The auxiliary space method and optimal multigrid preconditioning techniques for unstructured grids, Computing, Vol. 56, pp 215-235, 1996

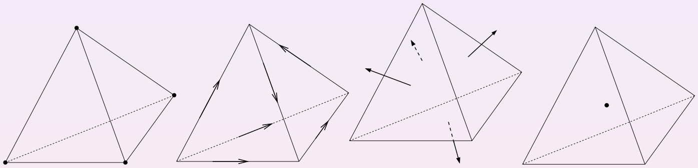


$W_1$: finite element space on a (auxiliary) structured grid.
Other relevant works:

*Nepomnyaschikh 1992, Bramble and Pasciak 1996, Chan, Smith and Zou 1996, Brenner 1996*

# Example 2: "far-away" auxiliary spaces

$$
\begin{array}{ccccccccccc}
R & \longrightarrow & C^\infty & \xrightarrow{\text{grad}} & C^\infty & \xrightarrow{\text{curl}} & C^\infty & \xrightarrow{\text{div}} & C^\infty & \longrightarrow & 0 \\
 & & \downarrow \Pi_h^{\text{grad}} & & \downarrow \Pi_h^{\text{curl}} & & \downarrow \Pi_h^{\text{div}} & & \downarrow \Pi_h^{0} & & \\
R & \longrightarrow & H_h(\text{grad}) & \xrightarrow{\text{grad}} & H_h(\text{curl}) & \xrightarrow{\text{curl}} & H_h(\text{div}) & \xrightarrow{\text{div}} & L_h^2 & \longrightarrow & 0
\end{array}
$$



Auxiliary space preconditioner for H(curl) system (Hiptmair and Xu 2006)

$$
B_h^{\text{curl}} = S_h^{\text{curl}} + \Pi_h^{\text{curl}} \begin{pmatrix} B_h^{\text{grad}} & 0 & 0 \\ 0 & B_h^{\text{grad}} & 0 \\ 0 & 0 & B_h^{\text{grad}} \end{pmatrix} (\Pi_h^{\text{curl}})^T + \text{grad}\ B_h^{\text{grad}}\ (\text{grad})^T
$$

Note: $\Pi = \text{grad}$ is a much more non-trivial than the usual "interpolation" operator.

## Auxiliary space preconditioner for H(div) system (Hiptmair and Xu 2006)

$$
\begin{array}{ccccccccccc}
R & \longrightarrow & C^\infty & \xrightarrow{\text{grad}} & C^\infty & \xrightarrow{\text{curl}} & C^\infty & \xrightarrow{\text{div}} & C^\infty & \longrightarrow & 0 \\
 & & \downarrow \Pi_h^{\text{grad}} & & \downarrow \Pi_h^{\text{curl}} & & \downarrow \Pi_h^{\text{div}} & & \downarrow \Pi_h^0 & & \\
R & \longrightarrow & H_h(\text{grad}\,) & \xrightarrow{\text{grad}} & H_h(\text{curl}\,) & \xrightarrow{\text{curl}} & H_h(\text{div}\,) & \xrightarrow{\text{div}} & L_h^2 & \longrightarrow & 0
\end{array}
$$

$$
\begin{aligned}
B_h^{\text{div}} &= S_h^{\text{div}} + \Pi_h^{\text{div}} \mathbf{B}_h^{\text{grad}} (\Pi_h^{\text{div}})^T \\
&+ \text{curl}\; S_h^{\text{curl}} (\text{curl}\,)^T + (\Pi_h^{\text{div}}\text{curl}\,)\mathbf{B}_h^{\text{grad}} (\Pi_h^{\text{div}}\text{curl}\,)^T
\end{aligned}
$$

Note: $\Pi = \text{curl}$ or $\Pi_h^{\text{div}}\text{curl}$ is a much more non-trivial than the usual “interpolation” operator.

# Outline


1. Prologue: overlapping domain decomposition method
2. Convergence analysis for overlapping DDM
3. The method of subspace correction: framework and theory
4. Method of alternating projection
5. Interpretation of MG as the method of subspace correction
6. On problems with strongly discontinuous jumps
7. Auxiliary space method — the Method of Auxiliary-space correction
8. Epilogue

# Concluding remarks

# Concluding remarks

- The Method of subspace correction (MSC) is a general algorithmic and theoretical framework of general iterative methods for (linear as well as nonlinear) problems.

# Concluding remarks

- The Method of subspace correction (MSC) is a general algorithmic and theoretical framework of general iterative methods for (linear as well as nonlinear) problems.
- Using MSC framework, relationship and difference between different algorithms can be easily understood.

# Concluding remarks

- The Method of subspace correction (MSC) is a general algorithmic and theoretical framework of general iterative methods for (linear as well as nonlinear) problems.
- Using MSC framework, relationship and difference between different algorithms can be easily understood.
- It provide a general designing guideline of iterative algorithms

# Concluding remarks

- The Method of subspace correction (MSC) is a general algorithmic and theoretical framework of general iterative methods for (linear as well as nonlinear) problems.
- Using MSC framework, relationship and difference between different algorithms can be easily understood.
- It provide a general designing guideline of iterative algorithms
- The method of alternating projection (MAP) is equivalent to an MSC (in its orthogonal component)

# Concluding remarks

- The Method of subspace correction (MSC) is a general algorithmic and theoretical framework of general iterative methods for (linear as well as nonlinear) problems.
- Using MSC framework, relationship and difference between different algorithms can be easily understood.
- It provide a general designing guideline of iterative algorithms
- The method of alternating projection (MAP) is equivalent to an MSC (in its orthogonal component)
- By (mathematically) reformulating the multigrid method into an MSC, a new type of convergence theory has been developed for multigrid convergence analysis that does not require elliptic regularity (which are essential in earlier theories) and can be applied to problems such as locally adapted grids and problems with rough coefficients

# Concluding remarks (cont.)

- Within MSC, robust iterative methods can be developed for nearly singular systems

# Concluding remarks (cont.)

- Within MSC, robust iterative methods can be developed for nearly singular systems
- For problems with strongly discontinuous coefficients,
  - Multigrid can be made robust with PCG
  - Exotic coarse spaces for nonoverlapping DD may be simplified

# Concluding remarks (cont.)

- Within MSC, robust iterative methods can be developed for nearly singular systems
- For problems with strongly discontinuous coefficients,
  - Multigrid can be made robust with PCG
  - Exotic coarse spaces for nonoverlapping DD may be simplified
- A more general framework: the method of auxiliary-space correction
  - auxiliary spaces can be quite different from the original space
  - (practical but theoretically provable) algebraic multigrid methods have been developed for various applications (for H(grad) [Grasedyck-Xu], H(curl)/H(div)[Hiptmair and Xu]).