Augmented Krylov Iterations in Domain Decomposition Methods

Y.L.Gurieva[1], V.P.Ilin[1,2], D.V.Perevozkin[1]

[1]Institute of Computational Mathematics and Mathematical Geophysics, SBRAS

[2]Novosibirsk State University

DD23 Conference, 06.07. 2015

References

1. Toselli, A., Widlund, O.: Domain Decomposition Methods – Algorithms and Theory. Springer, Berlin, Heidelberg (2005)
2. Chapman, A., Saad, Y.: Deflated and Augmented Krylov Subspace Technique. Numer. Linear Algebra Applic. 4(1), 43-66 (1997)
3. Gander, M.J., Halpern, L., Santugini, K.: A New Coarse Grid Correction for RAS/AS. Domain Decomposition Methods in Science and Engineering XXI, Lecture Notes in Computational Science and Engineering, Springer-Verlag, (2013)

References

4. Nataf F. Optimized Schwarz Methods. Lecture Notes in Comput. Sci. and Eng., 2009, 233-240.
5. Il'in, V.P.: Finite Difference and Finite Volume Methods for Elliptic Equations (in Russian). ICMMG Publisher, Novosibirsk (2001)
6. Il'in, V.P.: Finite Element Methods and Technologies (in Russian). ICMMG Publisher, Novosibirsk (2007)
7. Butyugin, D.S., Gurieva, Y.L., Il'in, V.P., Perevozkin, D.V., Petukhov, A.V.: Functionality and Algebraic Solvers Technologies in Krylov Library (in Russian). Vestnik YuUrGU. Series "Computational mathematics and informatics". 2(3), 92-105 (2013)

References

8. Siberian Supercomputer Centre, http://www2.sscc.ru
9. Intel® Math Kernel Library (Intel® MKL), http://software.intel.com/en-us/intel-mkl
10. Official page of Domain Decomposition Methods, http://www.ddm.org

Boundary Value Problem

$$Lu = f(\vec{r}), \quad \vec{r} \in \Omega; \qquad lu|_\Gamma = g,$$

Notations

$$\Omega = \bigcup_{q=1}^{P} \Omega_q, \quad \bar{\Omega} = \Omega \bigcup \Gamma, \quad \bar{\Omega}_q = \Omega_q \bigcup \Gamma_q,$$

$$\Gamma_q = \bigcup_{q' \in \omega_q} \Gamma_{q,q'}, \quad \Gamma_{q,q'} = \Gamma_q \bigcap \bar{\Omega}_{q'}, \quad q' \neq q,$$

Ω_0 – **external domain**, $\bar{\Omega}_0 = \Omega_0 \bigcup \Gamma$,

$\Gamma_{q,0} = \Gamma_q \bigcap \bar{\Omega}_0 = \Gamma_q \bigcap \Gamma$ – **external boundary of** Ω_q,

$\Delta_{q,q'} = \Omega_q \bigcap \Omega_{q'}$ – **overlapping**,

$\Gamma_{q,q'} = \Gamma_{q',q}$ – **non-overlapping** $(\Delta_{q,q'} = 0)$

Generalized Schwarz Decomposition

$$L u_q(\vec{r}) = f_q, \quad \vec{r} \in \Omega_q,$$
$$l_{q,q'}(u_q)\big|_{\Gamma_{q,q'}} = g_{q,q'} \equiv l_{q',q}(u_{q'})\big|_{\Gamma_{q',q}},$$
$$q' \in \omega_q, \quad l_{q,0} u_q|_{\Gamma_{q,0}} = g, \quad q = 1, ..., P,$$

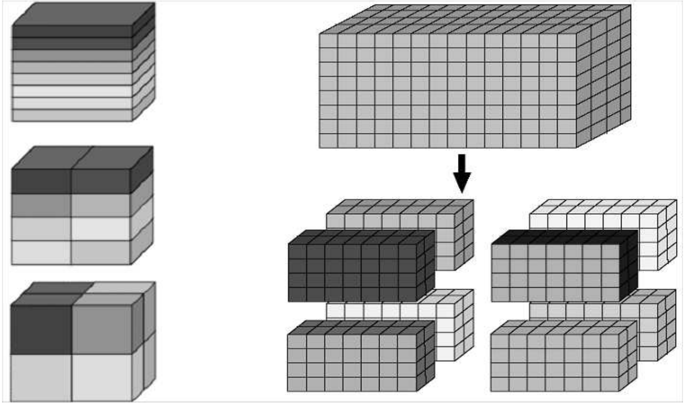
$$\alpha_q u_q + \beta_q \frac{\partial u_q}{\partial n_q}\Big|_{\Gamma_{q,q'}} = g_{q,q'} \equiv \alpha_{q'} u_q + \beta_{q'} \frac{\partial u_{q'}}{\partial n_{q'}}\Big|_{\Gamma_{q',q}},$$

$$|\alpha_q| + |\beta_q| > 0, \quad \alpha_q \cdot \beta_q \geq 0,$$

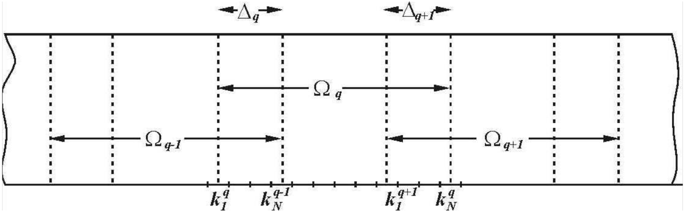
Iterations: $L u_q^n = f_q, \;\; l_{q,q'} u_q^n|_{\Gamma_{q,q'}} = l_{q',q} u_{q'}^{n-1}|_{\Gamma_{q',q}},$

$$?\; l_{q,q'} u_q = \alpha_q u_q + \beta_q \frac{\partial u_q}{\partial n_q} + \gamma_q \frac{\partial^2 u_q}{\partial \tau_q^2}$$

Examples of 1D-, 2D- and 3D- domain decomposition



1D-domain decomposition with overlapping



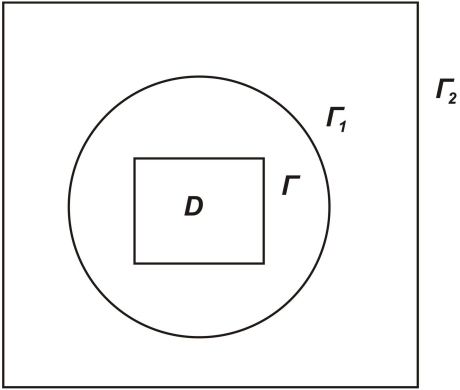
$$\Omega = \bigcup_{q=1}^{P} \Omega_q \text{ - computational domain,}$$

$$\Omega_q = \{k_1^q, ..., k_N^q\} \text{ - } q\text{-th subdomain,}$$

$$\Delta_q, \Delta_{q+1} \text{ - overlapping,}$$

$$k_1^q = k_N^{q-1},\ k_1^{q+1} = k_N^q \text{ - non-overlapping}$$

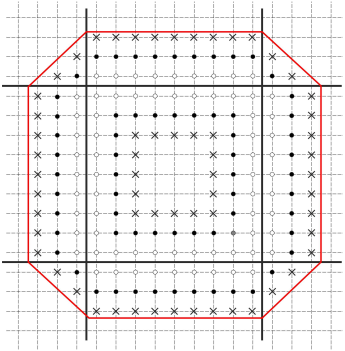
Combined Alternating Schwarz Method of Solving Exterior BVP by means of FEM and Integral Presentation



$$\Omega = R^3 / D$$

Automatical Construction of the Balanced Grid Subdomains

$$\bar{\Omega}_p \equiv \Omega_p \cup \Gamma_p^1 \ldots \cup \Gamma_p^{\Delta},$$



Algebraic Statement of the Problem

$$Au = \sum_{l' \in \omega_l} a_{l,l'} u_{l'} = f, \quad A = \{a_{l,l'}\} \in \mathcal{R}^{N,N},$$

$$u = \{u_l\}, \quad f = \{f_l\} \in \mathcal{R}^N,$$

$$\Omega = \{l\} = \bigcup_{s=1}^{P} \Omega_s, \quad N = \sum_{s=1}^{P} N_s,$$

$$\Gamma_s \equiv \Gamma_s^0 = \{l' \in \omega_l, \quad l \in \Omega_s, \quad l' \notin \Omega_s\}, \quad \bar{\Omega}_s^0 = \Omega_s \bigcup \Gamma_s^0,$$

$$\Gamma_s^t = \left\{ l' \in \omega_l, \quad l \in \bar{\Omega}_s^{t-1}, \quad \bar{\Omega}_s^t = \bar{\Omega}_s^{t-1} \bigcup \Gamma_s^t, \quad t = 1, 2, ..., \Delta_s \right\}$$

Additive Schwarz - Jacobi Preconditioners

$$\bar{u}_s = \{u_l, l \in \bar{\Omega}_s^{\Delta_s}\} \in \mathcal{R}^{\bar{N}_s}, \quad u = \bigcup_{s=1}^{P} \bar{u}_s,$$

$$A_{s,s}\bar{u}_s + \sum_{s' \in Q_s} A_{s,s'}\bar{u}_{s'} = f_s, \quad s = 1, ..., P,$$

$$\bar{B}_s(\bar{u}_s^{n+1} - \bar{u}_s^n) = \bar{f}_s - (\bar{A}\bar{u}^n)_s \equiv \bar{r}_s^n, \quad \bar{u}_s^n \in \mathcal{R}^{\bar{N}_s},$$

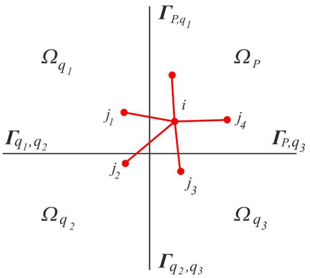
$$u_s^n = R_s\bar{u}_s^n = \{u_l^n = (R_s\bar{u}_s^n)_l, \quad l \in \Omega_s\} \in \mathcal{R}^{N_s},$$

RAS: $u^{n+1} = u^n + B_{ras}^{-1} r^n,$

$$B_{ras}^{-1} = R\hat{A}^{-1}W^T, \quad \hat{A} = W^T A W =$$

$$= \textbf{block-diag}\ \{A_{s,s} \in \mathcal{R}^{\bar{N}_s, \bar{N}_s}\}$$

Interface Conditions on the Internal Boundary



$$A_{s,s}\bar{u}_s + \sum_{s' \in Q_s} A_{s,s'}\bar{u}_{s'} = f_s, \quad s = 1, \ldots, P,$$

$$\left(a_{l,l} + \theta_l \sum_{l' \neq \Omega_s^{\Delta}} a_{l,l'}\right) u_l^n + \sum_{l \in \Omega_s^{\Delta}} a_{l,l'} u_{l'}^n =$$

$$= f_l + \sum_{l' \notin \Omega_s^{\Delta}} a_{l,l'} \left(\theta_l u_l^{n-1} - u_{l'}^{n-1}\right)$$

Coarse Grid Correction

$$\sum_{k=1}^{N_c} \varphi_k(x,y) = 1,\ Au = f,$$

$$u = \{u_{i,j} \approx u^c_{i,j} = \sum_{k=1}^{N_c} c_k \varphi_k(x_i, y_j)\} = \Phi\hat{u} + \psi,$$

$$\hat{u} = \{c_k\} \in \mathcal{R}^{N_c},\ \Phi = [\varphi_1 ... \varphi_{N_c}] \in \mathcal{R}^{N,N_c},$$

$$\hat{A}\hat{u} \equiv \Phi^T A \Phi \hat{u} = \Phi^T f - \Phi^T A\psi \equiv \hat{f} \in \mathcal{R}^{N_c},$$

$$\hat{A}\check{u} = \Phi^T f \equiv \check{f},$$

$$u \approx \tilde{u} = \Phi\check{u} = \Phi\hat{A}^{-1}\hat{f} = B_c^{-1} f,\ B_c^{-1} = \Phi(\Phi^T A \Phi)^{-1}\Phi^T$$

Deflated Conjugate Gradient Method

$$u^0 = u^{-1} + B_c^{-1} r^{-1}, \quad r^{-1} = f - Au^{-1},$$

$$\Phi^T r^0 = \Phi^T (r^{-1} - A\Phi \hat{A}^{-1} \Phi^T r^{-1}) = 0,$$

$$p^0 = (I - B_c^{-1} A) r^0, \; \Phi^T A p^0 = 0,$$

$$u^{n+1} = u^n + \alpha_n p^n, \quad r^{n+1} = r^n - \alpha_n A p^n,$$

$$p^{n+1} = r^{n+1} + \beta_n p^n - B_c^{-1} A r^{n+1},$$

$$\alpha_n = (r^n, r^n)/(p^n, Ap^n), \quad \beta_n = (r^{n+1}, r^{n+1})/(r^n, r^n),$$

$$n = 0, 1, \ldots : \quad \Phi^T r^{n+1} = 0, \; \Phi^T A p^{n+1} = 0$$

Multi-Preconditioned Semi-Conjugate Residual Method

$$P_0 = [p_1^0 \cdots p_m^0] \in \mathcal{R}^{N,m}, \quad p_l^0 = (B_0^{(l)})^{-1} r^0,$$

$$u^{n+1} = u^n + P_n \bar{\alpha}_n = u^0 + P_0 \bar{\alpha}_0 + \ldots + P_n \bar{\alpha}_n,$$

$$r^{n+1} = r^n - AP_n \bar{\alpha}_n = r^0 - AP_0 \bar{\alpha}_0 - \ldots - AP_n \bar{\alpha}_n,$$

$$P_n^T A^T A P_k = D_{n,k} = 0 \quad \textbf{for} \quad k \neq n,$$

$$\bar{\alpha}_n = (D_{n,n}^{-1})^{-1} P_n^T A^T r^0,$$

$$P_k^T A^T r^{n+1} = 0, \quad k = 0, 1, \ldots, n,$$

$$P_{n+1} = Q_{n+1} - \sum_{k=0}^{n} P_k \bar{\beta}_{k,n},$$

$$\bar{\beta}_{k,n} = D_{k,k}^{-1} P_k^T A^T A Q_{n+1},$$

$$Q_{n+1} = [q_1^{n+1} \ldots q_m^{n+1}], \quad q_l^{n+1} = (B_{n+1}^{(l)})^{-1} r^{n+1}$$

Two Dimensional Test Problem

$$-\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} + p\frac{\partial u}{\partial x} + q\frac{\partial u}{\partial y} = f(x,y), \quad (x,y) \in \Omega,$$

$$u|_\Gamma = g(x,y), \; \Omega = (a_x, b_x) \times (a_y, b_y)$$

$$x_i = a_x + ih_x, \quad y_j = a_y + jh_y,$$

$$i = 0, 1, \ldots, N_x + 1; \quad j = 0, 1, \ldots, N_y + 1;$$

$$h_x = (b_x - a_x)/(N_x + 1), \quad h_y = (b_y - a_y)/(N_y + 1),$$

$$\Omega = \bigcup_{s=1}^{P} \Omega_s, \quad P = P_x P_y,$$

PARDISO + BiCGStab, $\varepsilon = 10^{-8}$

Table 1 (2D). The numbers of iterations and the solution times (in seconds) on the grids 128^2 and 256^2 for different overlapping parameter Δ, $\varepsilon = 10^{-8}$

P	q	$N \setminus \Delta$	0		1		2		3		4		5	
	0		18	2.17	11	1.74	9	1.64	7	1.53	7	1.48	6	1.42
4	4	128	31	2.85	17	2.10	13	1.87	12	1.81	11	1.74	10	1.74
4	0	256	27	8.34	16	5.38	12	4.21	10	3.68	9	3.33	8	2.93
	4		61	16.88	25	7.74	19	6.52	17	5.47	15	5.28	13	4.25
	0		32	1.46	18	1.29	14	1.25	12	1.17	11	1.03	9	0.98
16	4	128	41	1.60	25	1.40	19	1.31	16	1.18	14	1.17	14	1.10
16	0	256	40	3.23	24	2.23	20	1.97	17	1.77	14	1.27	14	1.24
	4		58	4.32	35	2.83	28	2.46	22	1.98	19	1.62	18	1.52
	0		43	1.56	26	1.66	19	1.39	16	1.50	14	1.56	12	0.86
64	4	128	57	2.02	34	1.91	26	1.78	21	1.98	20	1.69	18	1.35
64	0	256	60	4.75	36	4.16	27	3.35	22	3.11	20	3.00	18	4.66
	4		87	7.04	47	5.61	38	4.89	31	4.13	28	4.02	25	4.48

Table 2 (2D). Aggregation influence (upper and low lines in the cells) in the additive Schwarz method (decomposition with different overlapping parameter $\Delta = 0, 1, 2$, see respected columns)

$N \setminus P$	2^2	4^2	8^2
64^2	19 11 8 23 9 7	26 15 12 21 12 9	37 20 15 28 15 11
128^2	29 15 11 24 14 10	35 22 17 26 16 12	51 31 21 36 21 15
256^2	38 21 17 31 18 15	53 31 23 35 21 17	71 43 32 40 26 21

3 D Test Problem

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} + p\frac{\partial u}{\partial x} + q\frac{\partial u}{\partial y} + r\frac{\partial u}{\partial z} = f(x, y, z),$$

$$(x, y, z) \in \Omega,\ u|_\Gamma = g(x, y, z),\ \Omega = [0; 1]^3,$$

$$u(x, y, z) = x^2 + y^2 + z^2,\ u^0 = 0,\ \varepsilon_{ex} = 10^{-7}$$

Monotone Exponential Fitting FVM on the Cubic Grid

Solver in Subdomains:

Eisenstat IF + FGMRES, $\varepsilon_{in} = 0.1$

P **Irregular Subdomains**

M **Coarse Grid Nodes**

Table 3 (3D). The numbers of iterations and total times, $p = q = r = 16$, $\Delta = 0$, $\Theta = 0$, $M = 30$.
SCR: Coarse Grid Corrections (CGC) every $m = 5$ iterations
BlSCR: Additive Schwarz + CGC, $m = 1$

P		4	8	16	32	64
32^3	scr	52 0.34	59 0.27	59 0.23	66 0.30	70 0.42
	blscr	45 0.48	54 0.34	54 0.32	62 0.38	67 0.48
64^3	scr	66 4.81	82 2.71	101 1.96	102 1.72	105 2.07
	blscr	59 5.35	70 3.18	85 2.39	98 2.32	109 2.66
128^3	scr	114 217.20	132 72.51	133 33.12	151 22.27	150 20.57
	blscr	101 226.26	111 79.05	134 43.15	156 32.79	159 30.71

Table 4 (3D). The numbers of iterations and total times, $p = q = r = 16$, $\Delta = 0$, $\Theta = 0$, $N = 128^3$.
SCR: Coarse Grid Corrections (CGC) every $m = 5$ iterations
BlSCR: Additive Schwarz + CGC, $m = 1$ $\Delta = 0, \Theta = 0$.

P	4		8		16		32	
fgmres	87	23.21	78	12.05	93	12.96	91	13.49
scr(m=20)	104	24.68	111	17.05	164	17.82	183	18.03
blscr(m=20)	96	52.58	106	31.78	128	26.07	126	23.18
scr(m=100)	87	32.75	95	19.99	119	15.79	131	15.59
blscr(m=100)	83	80.34	89	43.87	105	30.42	114	25.29