A TDNNS-FETI method for compressible and almost incompressible elasticity

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Overview

Overview

Linear elasticity

$\Omega \subset \mathbb{R}^d$, $d = 2$ or 3

$$\begin{aligned} -\mathbf{div}\underline{\boldsymbol{\sigma}} &= \mathbf{f} && \text{in } \Omega \\ \underline{\boldsymbol{\sigma}} &= \underline{\mathbf{D}}\,\underline{\varepsilon}(\mathbf{u}) && \text{in } \Omega, \quad \text{where } \underline{\varepsilon}(\mathbf{u}) = \tfrac{1}{2}(\nabla \mathbf{u} + \nabla^\top \mathbf{u}) \\ \mathbf{u} &= \mathbf{u}_D && \text{on } \Gamma_D \\ \boldsymbol{\sigma}_n &= \mathbf{t}_N && \text{on } \Gamma_N \end{aligned}$$

Hooke's law:

$$\underline{\mathbf{D}}\,\underline{\varepsilon} = \frac{E}{(1+\nu)}\underline{\varepsilon} + \frac{E\nu}{(1+\nu)(1-2\nu)}\operatorname{tr}(\underline{\varepsilon})I$$

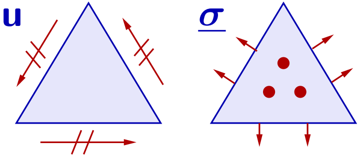
for $d = 2$: plane strain or plane stress

TDNNS Formulation

A. Pechstein & J. Schöberl '11, '12

Mixed TDNNS finite element (P^1, here 2D)

Tangential
continuous
Displacement
(Nédélec II)



symmetric
Normal-**N**ormal
continuous
Stress

Discrete variational formulation

$$\sum_T \int_T (\mathbf{div}\underline{\boldsymbol{\sigma}} + \mathbf{f}) \cdot \mathbf{v}\, dx + \sum_F \int_F \llbracket \sigma_{nt} \rrbracket \cdot \mathbf{v}_t\, ds = g_2 \qquad \forall \mathbf{v}$$

$$\sum_T \int_T (\mathbf{D}^{-1}\underline{\boldsymbol{\sigma}} - \underline{\varepsilon}(\mathbf{u})) : \underline{\boldsymbol{\tau}}\, dx + \sum_F \int_F \llbracket u_n \rrbracket \tau_{nn}\, ds = g_1 \qquad \forall \underline{\boldsymbol{\tau}}$$

Stability & Convergence:

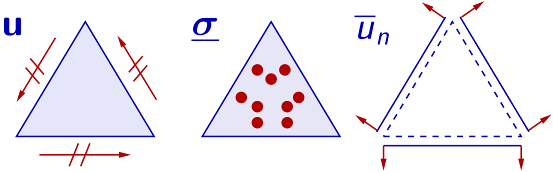
- Discrete inf-sup stability
- Optimal error estimates

But:

- Saddle point system

Hybridized TDNNS Formulation

Normal-normal continuity of stresses is broken and reenforced facet-wise by Lagrange multipliers, these approximate normal displacements.



Benefits:

- algebraically equivalent to previous system

Hybridized TDNNS Formulation

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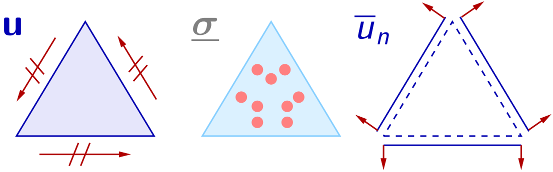


Benefits:

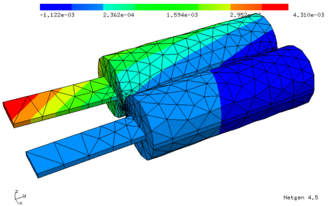
- algebraically equivalent to previous system
- stresses local $\rightsquigarrow$ static condensation
- condensed system is SPD

$$\underline{\mathbf{K}}\mathbf{U} = \mathbf{F} \qquad \mathbf{U} \,\widehat{=}\, \begin{bmatrix} \mathbf{u} \\ \overline{u}_n \end{bmatrix}$$

Why use TDNNS?

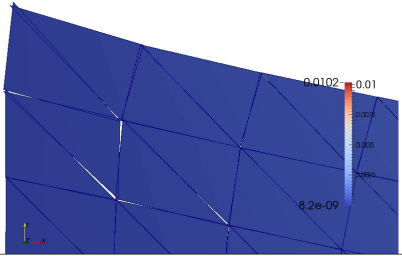
Single framework, stable w.r.t.

- volume locking (more later on)
- shear locking



DG-like method, lives up to the motto

Relax as much as you can, but never lose stability!



Overview

TDNNS (Tangential Displacement – Normal-Normal Stress)

Motivation of TDNNS-FETI

TDNNS-FETI Method

Theory

Numerical Results

Motivation I – Solver

Astrid's PhD thesis '09

Additive Schwarz preconditioner for high order (hybridized) system

- lowest order block (direct)
- overlapping blocks associated to edges, facets, elements
- optimal in h and polynomial degree

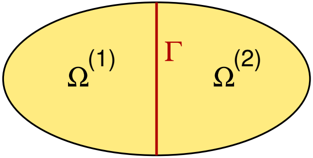
But up to now no scalable lowest order TDNNS-solver available!

Motivation II – Symbolic Study

Victorița Dolean, DD20 plenary talk

T. Cluzeau, V. Dolean, F. Nataf, and A. Quadrat '12

DD on PDE level



two subdomains, $\Gamma =$ symmetry axis

- scalar PDE $-\Delta u = f$
  coupling pair $(u, \frac{\partial u}{\partial \mathbf{n}})$
  Neumann-Neumann preconditioner is exact
- elasticity $-\mathbf{div}(\underline{\boldsymbol{\sigma}}(\mathbf{u})) = \mathbf{f}$
  coupling pair $(\mathbf{u}, \boldsymbol{\sigma}_n(\mathbf{u}))$
  ?

Motivation II – Symbolic Study

Victorița Dolean, DD20 plenary talk — DD on PDE level

T. Cluzeau, V. Dolean, F. Nataf, and A. Quadrat '12

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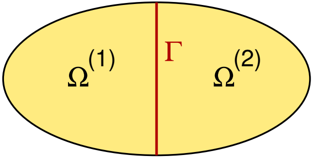
two subdomains, Γ = symmetry axis



- scalar PDE $-\Delta u = f$
  coupling pair $(u, \frac{\partial u}{\partial \mathbf{n}})$
  Neumann-Neumann preconditioner is exact
- elasticity $-\mathbf{div}(\underline{\boldsymbol{\sigma}}(\mathbf{u})) = \mathbf{f}$
  coupling pair $(\mathbf{u}, \boldsymbol{\sigma}_n(\mathbf{u}))$
  Neumann-Neumann preconditioner is NOT exact!

Motivation II – Symbolic Study

Victorița Dolean, DD20 plenary talk

T. Cluzeau, V. Dolean, F. Nataf, and A. Quadrat '12

DD on PDE level

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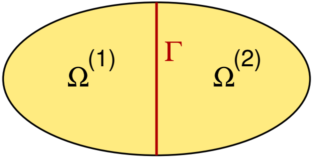
two subdomains, Γ = symmetry axis



- scalar PDE $-\Delta u = f$
  coupling pair $(u, \frac{\partial u}{\partial \mathbf{n}})$
  Neumann-Neumann preconditioner is exact
- elasticity $-\mathbf{div}(\underline{\boldsymbol{\sigma}}(\mathbf{u})) = \mathbf{f}$
  coupling pair $\left(\begin{bmatrix} \mathbf{u}_t \\ \sigma_{nn}(\mathbf{u}) \end{bmatrix}, \begin{bmatrix} \boldsymbol{\sigma}_{nt}(\mathbf{u}) \\ u_n \end{bmatrix} \right)$
  Neumann-Neumann preconditioner is exact

symbolic approach

Smith factorization

`OreMorphisms`

Existing Work – FETI for mechanics

Compressible elasticity

- Farhat & Roux '94 etc. – FETI (original)
- Farhat, Lesoinne, LeTallec, Pierson & Rixen '01 – FETI-DP (original)
- Dostál, Horák & R. Kučera '06 – TFETI
- Klawonn & Widlund '06 – FETI-DP (3D coarse space)
- Klawonn, Rheinbach & Widlund '08 – FETI-DP (2D, irregular interface)

Incompressible Stokes

- Li & Widlund '06 – BDDC (primal pressure dofs, quasi-optimal)
- Kim & Lee '10 – FETI-DP (no primal pressure dofs, suboptimal)

Almost incompressible elasticity

- Dohrmann & Widlund '09 – OS ("full" coarse space)
- Dohrmann & Widlund '09 – hybrid OS-IS (reduced coarse space)
- Pavarino, Widlund & Zampini '10 – BDDC (reduced coarse space)
- Gippert, Rheinbach & Klawonn '12 – FETI-DP unresolved incompressible inclusions

Your paper here
& even more at this conference!

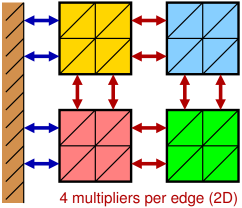
We won't compare TDNNS-FETI with the above FETI(-DP) methods
but show what one can do when one has chosen TDNNS as discretization.

Overview

one-level (T)FETI for hybridized TDNNS

Saddle point formulation

$$\begin{bmatrix} \underline{\mathbf{K}}^{(1)} & & 0 & \underline{\mathbf{B}}^{(1)\top} \\ & \ddots & & \vdots \\ 0 & & \underline{\mathbf{K}}^{(N)} & \underline{\mathbf{B}}^{(N)\top} \\ \underline{\mathbf{B}}^{(1)} & \dots & \underline{\mathbf{B}}^{(N)} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{U}^{(1)} \\ \vdots \\ \mathbf{U}^{(N)} \\ \boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} \mathbf{F}^{(1)} \\ \vdots \\ \mathbf{F}^{(N)} \\ 0 \end{bmatrix}$$



Kernel property

$$\ker(\underline{\mathbf{K}}^{(i)}) = \operatorname{range}(\underline{\mathbf{R}}^{(i)})$$ rigid body modes

Elimination of $\mathbf{U}^{(i)}$

by generalized inverse $\underline{\mathbf{K}}^{(i)\dagger}$

$$\begin{bmatrix} \underline{\mathbf{B}}\,\underline{\mathbf{K}}^{\dagger}\underline{\mathbf{B}} & -\underline{\mathbf{G}} \\ \underline{\mathbf{G}}^{\top} & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{\lambda} \\ \xi \end{bmatrix} = \begin{bmatrix} \mathbf{d} \\ \mathbf{e} \end{bmatrix}$$

$$\underline{\mathbf{G}} := \underline{\mathbf{B}}\,\underline{\mathbf{R}}$$
$$\mathbf{U} = \underline{\mathbf{K}}^{\dagger}(\mathbf{F} - \underline{\mathbf{B}}^{\top}\boldsymbol{\lambda}) + \underline{\mathbf{R}}\xi$$

Projection method

$$\underline{\mathbf{P}}_{\underline{\mathbf{Q}}} := \underline{\mathbf{I}} - \underline{\mathbf{Q}}\,\underline{\mathbf{G}}(\underline{\mathbf{G}}^{\top}\underline{\mathbf{Q}}\,\underline{\mathbf{G}})^{-1}\underline{\mathbf{G}}^{\top}$$

One-level (T)FETI with scaled Dirichlet preconditioner

Solve (PCG) $\underline{\mathbf{P}_{\mathbf{Q}}}^\top \underline{\mathbf{B}}\, \underline{\mathbf{K}}^\dagger \underline{\mathbf{B}}^\top \widetilde{\boldsymbol{\lambda}} = \underline{\mathbf{P}_{\mathbf{Q}}}^\top \widetilde{\mathbf{d}}$ $\quad \widetilde{\boldsymbol{\lambda}} \in \mathrm{range}(\underline{\mathbf{P}_{\mathbf{Q}}})$

Preconditioner $\underline{\mathbf{P}_{\mathbf{Q}}}\ \underline{\mathbf{B}}_D \underline{\mathbf{S}}\, \underline{\mathbf{B}}_D^\top$ $\quad \underline{\mathbf{S}} = \mathrm{diag}(\underline{\mathbf{S}}^{(i)})$

Yet to be specified:

- scalings in $\underline{\mathbf{B}}_D = [\underline{\mathbf{D}}^{(1)} \underline{\mathbf{B}}^{(1)} | \ldots | \underline{\mathbf{D}}^{(N)} \underline{\mathbf{B}}^{(N)}]$
- matrix $\underline{\mathbf{Q}}$

Choice of Scalings – 2D

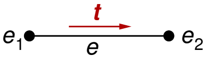
Scalings in $\underline{\mathbf{B}}_D$

diagonal entry of $\underline{\mathbf{D}}^{(i)}$ at multiplier linking (i,j)		
$\frac{1}{2}$ multiplicity scaling	$\frac{E^{(i)}}{E^{(i)}+E^{(j)}}$ coefficient scaling	$\frac{K^{(i)}_{\text{diag,max}}}{K^{(i)}_{\text{diag,max}}+K^{(j)}_{\text{diag,max}}}$ stiffness scaling

Choices of $\underline{\mathbf{Q}}$

- $\underline{\mathbf{Q}} = \underline{\mathbf{I}}$
- $\underline{\mathbf{Q}} = \underline{\mathbf{M}}^{-1} = \underline{\mathbf{B}}_D \underline{\mathbf{S}}\, \underline{\mathbf{B}}_D^\top$
- $(\underline{\mathbf{Q}}_{\text{diag}})_{m,ij} = \frac{\min(E^{(i)}, E^{(j)})}{h^{(ij)} H^{(ij)}}$ or using scaled stiffness entries

Assumption on degrees of freedom



$$\int_e \mathbf{v}\cdot\mathbf{t}\, ds \qquad \frac{1}{|e|}\int_e (\mathbf{v}\cdot\mathbf{t}) s\, ds \qquad |e|\, \overline{v}_{n|e}(e_1) \qquad |e|\, \overline{v}_{n|e}(e_2)$$

Overview

TDNNS (Tangential Displacement – Normal-Normal Stress)

Motivation of TDNNS-FETI

TDNNS-FETI Method

Theory

Numerical Results

Condition Number Estimate

Assumptions

- $d = 2$
- triangular mesh — technical
- $E_{|\Omega^{(i)}} \approx E^{(i)} = \text{const}$
- subdomains = compounds of coarse elements — can be relaxed
- subdomain meshes quasi-uniform — can be relaxed

Theorem (AP & CP)

For coefficient or stiffness scaling, and $\underline{\mathbf{Q}} = \underline{\mathbf{M}}^{-1}$ *or* $\underline{\mathbf{Q}} = \underline{\mathbf{Q}}_{diag}$

$$\kappa_{(T)FETI} \;\leq\; C\,(1 + \log(\tfrac{H}{h}))^2$$

C *independent of* H, h, *and jumps in* E.

Discrete norms

Energy norm is equivalent to the DG-like (semi)norm

$$|\mathbf{v}, \overline{v}_n|^2_{\mathbf{H}(\underline{\varepsilon},\Omega,\mathcal{T}_h)} := \sum_T \left(\|\underline{\varepsilon}(\mathbf{v})\|^2_{\underline{\mathbf{L}}^2(T)} + h_T^{-1} \|\mathbf{v}\cdot\mathbf{n} - \overline{v}_n\|^2_{L^2(\partial T)} \right)$$

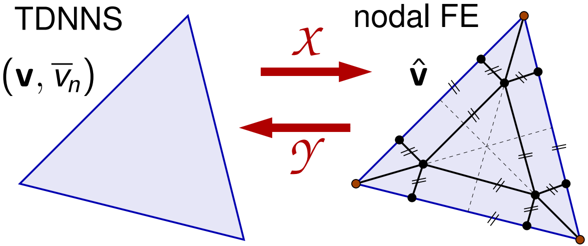
or (via Korn-type inequality, Brenner '04)

$$|\mathbf{v}, \overline{v}_n|^2_{\mathbf{H}^1(\Omega,\mathcal{T}_h)} := \sum_T \left(|\mathbf{v}|^2_{\underline{\mathbf{H}}^1(T)} + h_T^{-1} \|\mathbf{v}\cdot\mathbf{n} - \overline{v}_n\|^2_{L^2(\partial T)} \right)$$

"L^2-norm"

$$\|\mathbf{v}, \overline{v}_n\|^2_{\mathbf{L}^2(\Omega,\mathcal{T}^h)} := \|\mathbf{v}\|^2_{\mathbf{L}^2(\Omega)} + \sum_T h \, \|\overline{v}_n\|^2_{L^2(\partial T)}$$

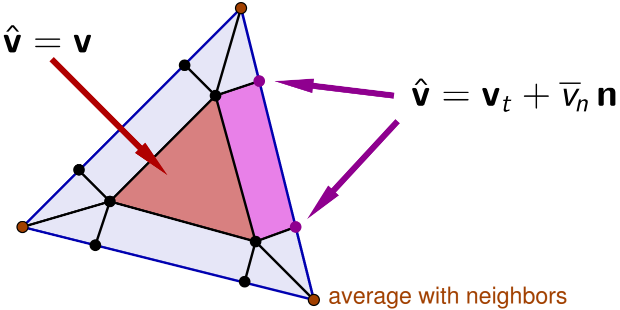
Two Transformations



Features

- preserve continuous functions, in particular rigid body motions
- H^1-stable $\quad |\mathcal{X}(\mathbf{v}, \overline{v}_n)|_{\mathbf{H}^1(\Omega)} \lesssim |\mathbf{v}, \overline{v}_n|_{\mathbf{H}^1(\Omega,\mathcal{T}_h)} \qquad |\mathcal{Y}\widehat{\mathbf{v}}|_{\mathbf{H}^1(\Omega,\mathcal{T}_h)} \lesssim |\mathbf{v}|_{\mathbf{H}^1(\Omega)}$
- L^2-stable $\quad \|\mathcal{X}(\mathbf{v}, \overline{v}_n)\|_{\mathbf{L}^2(\Omega)} \lesssim \|\mathbf{v}, \overline{v}_n\|_{\mathbf{L}^2(\Omega,\mathcal{T}_h)} \qquad \|\mathcal{Y}\widehat{\mathbf{v}}\|_{\mathbf{L}^2(\Omega,\mathcal{T}_h)} \lesssim \|\mathbf{v}\|_{\mathbf{L}^2(\Omega)}$
- $\lesssim$ involves only shape-regularity constants
- also works subdomain-wise / patch-wise

Main Idea of constructing $\hat{\mathbf{v}} := \mathcal{X}(\mathbf{v}, \overline{v}_n)$



$$|\mathbf{v}, \overline{v}_n|^2_{\mathbf{H}^1(\Omega,\mathcal{T}_h)} = \sum_T |\mathbf{v}|^2_{\mathbf{H}^1(T)} + h_T^{-1} \|\mathbf{v}\cdot\mathbf{n} - \overline{v}_n\|^2_{L^2(\partial T)}$$

Lift of Existing Tools

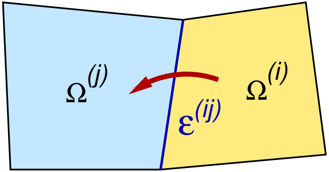
Cut-off Estimate (Edge Lemma)

$$|\Theta_{\mathcal{E}^{(ij)}}(\mathbf{v}, \overline{v}_n)|^2_{\mathbf{H}^1(\Omega^{(i)},\mathcal{T}_h)} \lesssim (1 + \log(\frac{H^{(i)}}{h^{(i)}}))^2 \, \|\mathbf{v}, \overline{v}_n\|^2_{\mathbf{H}^1(\Omega^{(i)})}$$

Discrete Subdomain Extension

$$|\mathbb{E}^{(i\to j)}(\mathbf{v}, \overline{v}_n)|^2_{\mathbf{H}^1(\Omega^{(j)},\mathcal{T}_h)} \lesssim |\mathbf{v}, \overline{v}_n|^2_{\mathbf{H}^1(\Omega^{(i)},\mathcal{T}_h)}$$

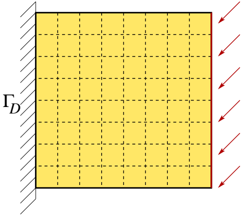
$$\|\mathbb{E}^{(i\to j)}(\mathbf{v}, \overline{v}_n)\|^2_{\mathbf{L}^2(\Omega^{(j)},\mathcal{T}_h)} \lesssim \|\mathbf{v}, \overline{v}_n\|^2_{\mathbf{L}^2(\Omega^{(i)},\mathcal{T}_h)}$$



Overview

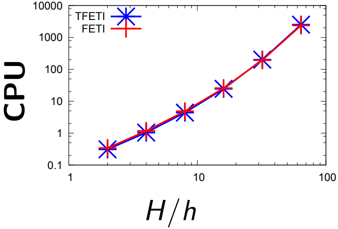
1) Compressible Case, Homogeneous Material

plane strain, $\nu = 0.3$, $E = 2 \cdot 10^5$ N/mm^2



64 subdomains, multiplicity scaling, $\underline{\mathbf{Q}} = \underline{\mathbf{I}}$

	FETI		TFETI	
H/h	cond	iter	cond	iter
2	15.37	22	3.24	15
4	19.82	28	5.34	20
8	25.00	34	8.43	25
16	30.03	38	11.84	29
32	35.19	42	15.59	33
64	40.65	47	19.65	36
	168 coarse dofs		192 coarse dofs	



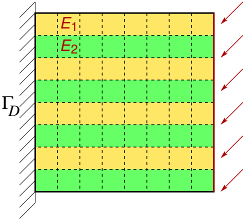
(sequential)

PCG tolerance 10^{-8} (relative)

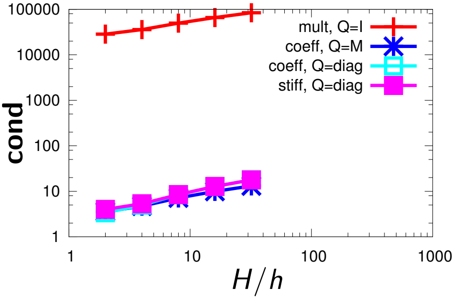
2) Compressible Case, Heterogeneous Material

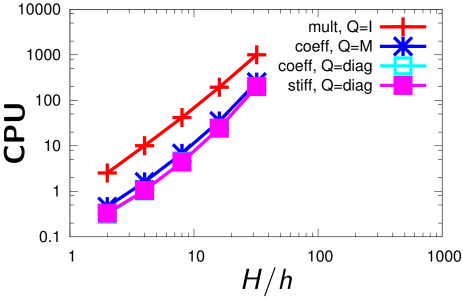
$\nu = 0.3$

$E_1 = 10^1$ N/mm^2 $\quad E_2 = 2 \cdot 10^5$ N/mm^2



TFETI (similar for FETI)





Almost Incompressible Case – Stabilization

Consistent stabilization

Astrid's PhD thesis '09

$$\sum_T \int_T (\mathbf{div}\underline{\boldsymbol{\sigma}} + \mathbf{f}) \cdot \mathbf{v}\, dx \;+\; \sum_F \int_F [\![\boldsymbol{\sigma}_{nt}]\!] \cdot \mathbf{v}_t\, ds \qquad = \quad g_2 \qquad \forall \mathbf{v}$$

$$\sum_T \int_T (\underline{\mathbf{D}}^{-1}\underline{\boldsymbol{\sigma}} - \underline{\varepsilon}(\mathbf{u})) : \underline{\boldsymbol{\tau}}\, dx \;+\; \sum_F \int_F \tau_{nn} [\![u_n]\!]\, ds \;+$$

$$+ \sum_T h_T^2 \int_T (\mathbf{div}\underline{\boldsymbol{\sigma}} + \mathbf{f}) \cdot \mathbf{div}\underline{\boldsymbol{\tau}}\, dx \quad = \quad g_1 \qquad \forall \underline{\boldsymbol{\tau}}$$

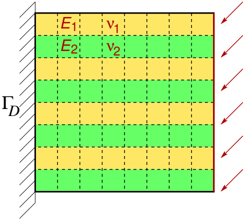
In hybridized form $\rightsquigarrow$ penalizes volume changes: $+\dfrac{\lambda}{|T|}\Big|\displaystyle\int_{\partial T} \overline{v}_n\, ds\Big|^2$

Stability & Convergence uniform for $\nu \to 1/2$

3) Almost Incompressible Case

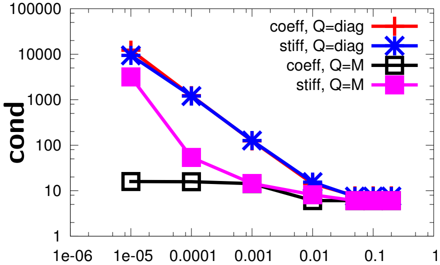
$\nu_1 \in [0.3,\ 0.49999]$ $\quad \nu_2 = 0.3$

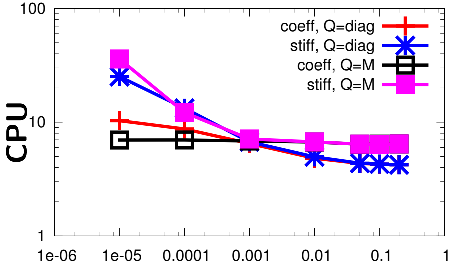
$E_1 = 10^1\ \mathrm{N/mm^2}$ $\quad E_2 = 2 \cdot 10^5\ \mathrm{N/mm^2}$



TFETI (similar for FETI)

$H/h = 8$





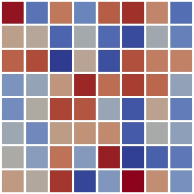
$\frac{1}{2} - \nu_1$ $\quad \frac{1}{2} - \nu_1$

4) Irregular Subdomains

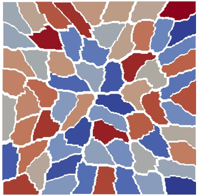
$\nu = 0.3 \qquad E = 2 \cdot 10^5$ N/mm^2

64 subdomains, multiplicity scaling, $\underline{\mathbf{Q}} = \underline{\mathbf{I}}$ $\qquad H/h \approx 32$

regular	FETI		TFETI	
	cond	iter	cond	iter
cont. P1	23.8	31	7.0	23
TDNNS	35.2	42	15.6	33



METIS	FETI		TFETI	
	cond	iter	cond	iter
cont. P1	75.2	49	34.1	43
TDNNS	113.6	67	90.7	64



Summary

One-level (T)FETI method for hybridized TDNNS discretization

- Formulation, choice of scalings (2D)
- Analysis for 2D compressible case
- Numerical results reveal good parameter choices for $\nu \to 1/2$

Benefits

- Very simple coarse space (RBM)
- Usual robustness properties
- Additional robustness for $\nu \to 1/2$

Ongoing work

- Analysis of incompressible case
- 3D
- Deluxe scaling
- Anisotropic domains

References

- A. Pechstein & C. Pechstein
  A FETI method for a TDNNS discretization of plane elasticity
  www.numa.uni-linz.ac.at/publications/List/2013/2013-05.pdf
- Astrid Sinwel. A New Family of Mixed Finite Elements for Elasticity
  www.numa.uni-linz.ac.at/Teaching/PhD/Finished/sinwel-diss.pdf
- A. Pechstein & J. Schöberl. Tangential-displacement and normal-normal stress continuous mixed finite elements for elasticity
  Math. Models Methods Appl. Sci. **21**(8):1761–1782, 2011
- A. Pechstein & J. Schöberl. Anisotropic mixed finite elements for elasticity
  Internat. J. Numer. Methods Engrg. **90**(2):196–217, 2012
- S. C. Brenner. Korn's inequalities for piecewise H^1 vector fields
  SIAM J. Numer. Anal. **41**(1):306–324, 2003

Thanks for your attention!