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# Mesoscopic and Macroscopic Mixed Variational Analysis of Two-Phase Flow in Fractured Porous Media

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## Part I: Mesoscopic Fractional Flow Models

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**6** Dual evolution macro-hybrid variational formulation

# 1 Physical Two-Phase Flow Model

- *Mixed velocity-pressure-saturation governing equations*

$$\left.\begin{aligned} \boldsymbol{u}_\alpha &= -\frac{\kappa_{r\alpha}(s_\alpha)}{\mu_\alpha}\boldsymbol{K}(\boldsymbol{grad}\, p_\alpha - \rho_\alpha \boldsymbol{g}) \\ \phi\frac{\partial(\rho_\alpha s_\alpha)}{\partial t} &+ div(\rho_\alpha \boldsymbol{u}_\alpha) = \rho_\alpha \hat{q}_\alpha(s_\alpha) \end{aligned}\right\} \text{ in } \Omega \times (0,T) \tag{1}$$

- *Flow volume balance constraint - Capillary pressure*

$$\begin{aligned} s_w + s_n &= 1 \\ p_c(s_w) &= p_n - p_w \end{aligned} \tag{2}$$

# 2 Fractional Two-Phase Flow Model

[Arbogast 1992, Chen-Ewing 1999]

- *Total velocity - Global pressure - Complementary pressure*

$$\boldsymbol{u} = \boldsymbol{u}_w + \boldsymbol{u}_n$$

$$p = p_n - \int_0^{s_w} \Big(f_w \frac{\partial p_c}{\partial s_w}\Big)(s)\; ds \tag{3}$$

$$\theta = -\int_0^{s_w} \Big(f_w f_n \frac{\partial p_c}{\partial s_w}\Big)(s)\; ds$$

- *Fractional flow mixed system*

$$\left.\begin{aligned}
&\lambda^{-1}(\theta)\boldsymbol{K}^{-1}\boldsymbol{u} = -\boldsymbol{grad}\; p - \boldsymbol{\gamma}_1(\theta) \\
&div\; \boldsymbol{u} = \hat{q}(\theta); \\
&\lambda^{-1}(\theta)\boldsymbol{K}^{-1}\boldsymbol{u}_w = -\boldsymbol{grad}\; \theta - f_w(\theta)\boldsymbol{grad}\; p - \boldsymbol{\gamma}_2(\theta) \\
&\vartheta(\theta)\frac{\partial \theta}{\partial t} + div\; \boldsymbol{u}_w = \hat{q}_w(\theta)
\end{aligned}\right\}\; \text{in}\; \Omega\times(0,T) \tag{4}$$

# 3 Instantaneous Total Velocity-Global Pressure Mixed Variational Model

[Alduncin 2005]

- *Subdifferential boundary conditions*

$$\begin{aligned} &\gamma_N p \in \partial\psi_N(\delta_N \boldsymbol{u}) = \partial I_{\{\widehat{u}\}}(\delta_N \boldsymbol{u}) \text{ in } B^*(\partial\Omega_N) \\ &\gamma_D p \in \partial\psi_D(\delta_D \boldsymbol{u}) = \{\widehat{p}\} \qquad\qquad \text{in } B^*(\partial\Omega_D) \end{aligned} \tag{5}$$

- *Variational Green Formula*

$$\boldsymbol{div}^T q + \boldsymbol{grad}\; q = \boldsymbol{\delta}^T \boldsymbol{\gamma}\; q \text{ in} \boldsymbol{V}^*(\Omega) \tag{6}$$

- *Fundamental trace compatibility property*

$$(\boldsymbol{C}_\delta) \quad \delta \in \boldsymbol{\mathcal{L}}(\boldsymbol{V}(\Omega), B(\partial\Omega)) \text{ is surjective}$$

- *Mixed variational model*

$$(\boldsymbol{M}) \begin{cases} \text{Find } \boldsymbol{u} \in \boldsymbol{V}(\Omega) \text{ and } p \in Y(\Omega) \\ \boldsymbol{div}^T p \in (\lambda(\theta(t))\boldsymbol{K})^{-1}\boldsymbol{u} + \boldsymbol{\partial}(I_{\{\widehat{u}(t)\}} \circ \delta_N)(\boldsymbol{u}) - \boldsymbol{f}^*_{\boldsymbol{\theta}}(t) \;\; \text{in } \boldsymbol{V}^*(\Omega) \\ -div\; \boldsymbol{u} \in \partial 0_Y(p) - \widehat{q}(t) \qquad\qquad \text{in } Y^*(\Omega) \end{cases}$$

- *Mixed functional framework*

$$\begin{aligned}
&\boldsymbol{V}(\Omega) = \boldsymbol{H}(div;\Omega) \equiv \{\boldsymbol{v} \in \boldsymbol{L}^2(\Omega) : div\ \boldsymbol{v} \in L^2(\Omega)\} \\
&Y(\Omega) = L^2(\Omega) \\
&B(\partial\Omega) = H^{-1/2}(\partial\Omega) \\
&B^*(\partial\Omega) = H^{1/2}(\partial\Omega)
\end{aligned} \tag{7}$$

- *Classical compatibility condition*

$$(\boldsymbol{C_{div}})\ div \in \boldsymbol{\mathcal{L}}(\boldsymbol{V}(\Omega), Y^*(\Omega)) \text{ is surjective}$$

- *Primal composition duality principle*

**Theorem 1** *Mixed problem* $(\boldsymbol{M})$ *is uniquely solvable if and only if its instantaneous variational primal problem*

$$(\boldsymbol{P}) \begin{cases} Find\ \boldsymbol{u} \in \boldsymbol{V}(\Omega) \\ \boldsymbol{0} \in (\lambda(\theta(t))\boldsymbol{K})^{-1}\boldsymbol{u} + \partial I_{\mathcal{N}(div)}(\boldsymbol{u} - \boldsymbol{u}_{\widehat{q}(t)}) + \boldsymbol{\partial}(I_{\{\widehat{u}(t)\}} \circ \delta_N)(\boldsymbol{u}) - \boldsymbol{f}^*_{\boldsymbol{\theta}}(t) \\ \hfill in\ \boldsymbol{V}^*(\Omega) \end{cases}$$

*is uniquely solvable, where* $div\ \boldsymbol{u}_{\widehat{q}(t)} = \widehat{q}(t)$

# 4 Evolution Wetting Velocity-Complementary Pressure Mixed Variational Model

[Alduncin 2007]

- *Evolution mixed functional Hilbert framework*

$$\boldsymbol{\mathcal{V}} = L^2(0,T;\boldsymbol{V}(\Omega)) = \{\boldsymbol{v} : (0,T) \to \boldsymbol{V}(\Omega) | \|\boldsymbol{v}\|_{\mathcal{V}} = [\textstyle\int_0^T \|\boldsymbol{v}(t)\|^2_{V\Omega)} dt]^{1/2} < \infty\}$$

$$\mathcal{Y} = L^2(0,T;Y(\Omega)) = \{y : (0,T) \to Y(\Omega) | \|y\|_{\mathcal{Y}} = [\textstyle\int_0^T \|y(t)\|^2_{Y\Omega)} dt]^{1/2} < \infty\} \tag{8}$$

- *Mixed variational model*

$$(\boldsymbol{\mathcal{M}}) \begin{cases} \text{Find } \boldsymbol{u_w} \in \boldsymbol{\mathcal{V}} \text{ and } \theta \in \mathcal{X} \\ \boldsymbol{div}^T \theta \in \boldsymbol{\partial F}(\boldsymbol{u_w}) - \boldsymbol{h^*_\theta} \quad \text{in } \boldsymbol{\mathcal{V}}^* \\ -div\ \boldsymbol{u_w} = A(\theta)\dfrac{\partial \theta}{\partial t} - \hat{q}_w \ \text{ in } \mathcal{Y}^* \\ \theta(0) = \widehat{\theta}_0 \end{cases}$$

- *Dual evolution duality principle*

**Theorem 2** *Dual evolution mixed problem* $(\boldsymbol{\mathcal{M}})$ *possesses a unique solution if and only if its dual evolution nonlinear problem*

$$(\boldsymbol{\mathcal{D}}) \begin{cases} \textit{Find } \theta \in \mathcal{X} \\ \\ 0 \in A(\theta)\dfrac{\partial \theta}{\partial t} + \partial(F^* \circ \boldsymbol{div}^T)(\theta + r^*_{h^*_\theta}) - \hat{q}_w \quad \textit{in } \mathcal{Y}^* \\ \\ \theta(0) = \hat{\theta}_0 \end{cases}$$

*possesses a unique solution, where* $\boldsymbol{div}^T r^*_{h^*_\theta} = \boldsymbol{h^*_\theta}$

- *Dual evolution solavability*

[Chen 2001]

**Theorem 3** *Dual evolution mixed problem* $(\boldsymbol{\mathcal{D}})$ *is uniquely solvable*

# 5 Dual Instantaneous Macro-Hybrid Variational Formulation

[Alduncin 2007]

- *Flow region nonoverlapping domain decomposition*

$$\overline{\Omega} = \bigcup_{e=1}^{E} \overline{\Omega}_e \qquad \Omega_e \cap \Omega_f = \emptyset \quad 1 \le e < f \le E$$

$$\Gamma_e = \partial\Omega_e \cap \Omega \quad 1 \le e \le E \tag{9}$$

$$\Gamma_{ef} = \Gamma_e \cap \Gamma_f \quad 1 \le e < f \le E$$

- *Specific fractured media decomposition*

$$\{\Omega_e\}_{e=1}^{E_R} \text{ and } \{\Omega_e\}_{e=E_R+1}^{E} \quad E = E_R + E_F \tag{10}$$

- *Decomposable total velocity and global pressure Hilbert framework*

$$\boldsymbol{V}(\Omega) \simeq \left\{ \boldsymbol{v} \in \boldsymbol{V}(\{\Omega_e\}) \equiv \prod_{e=1}^{E} \boldsymbol{V}(\Omega_e) : \{\delta_{\Gamma_e}\boldsymbol{v}\} \in \boldsymbol{Q} \right\}$$

$$Y(\Omega) \simeq \boldsymbol{Y}(\{\Omega_e\}) \equiv \prod_{e=1}^{E} Y(\Omega_e) \tag{11}$$

- *Fundamental macro-hybrid compatibility condition*

$$(\boldsymbol{C}_{[\delta_{\Gamma_e}]}) \quad [\delta_{\Gamma_e}] \in (\boldsymbol{\mathcal{L}}(\boldsymbol{V}(\{\Omega_e\}), \boldsymbol{B}(\{\Gamma_e\})) \text{ is surjective}$$

• *Primal and dual internal boundary transmission subspaces*

$$\boldsymbol{Q} \subset \boldsymbol{B}(\{\Gamma_e\}) = H^{1/2}(\{\Gamma_e\})$$
$$\boldsymbol{Q}^* = \Big\{\{\mu_e^*\} \in \boldsymbol{B}^*(\{\Gamma_e\}) : \ \langle\{\mu_e^*\}, \{\mu_e\}\rangle_{B(\{\Gamma_e\})} = 0, \forall\{\mu_e\} \in \boldsymbol{Q}\Big\} \tag{12}$$

**Lemma 4** *Due to* $(\boldsymbol{C}_{[\delta_{\Gamma_e}]})$*, macro-hybrid compositional dualization holds true*

$$\{\delta_{\Gamma_e}^T \lambda_e^*\} \in \boldsymbol{\partial}(I_Q \circ [\delta_{\Gamma_e}])(\{\boldsymbol{u_e}\}) \Longleftrightarrow \{\delta_{\Gamma_e}\boldsymbol{u_e}\} \in \boldsymbol{\partial I}_{Q^*}(\{\lambda_e^*\}) \tag{13}$$

• *Instantaneous macro-hybrid dual mixed problem*

$$(\boldsymbol{MH}) \begin{cases} \text{Find } \{\boldsymbol{u_e}\} \in \boldsymbol{V}(\{\Omega_e\}) \text{ and } \{p_e\} \in \boldsymbol{Y}(\{\Omega_e\}) \\ \{\boldsymbol{div}^T p_e\} - \{\delta_{\Gamma_e}^T \lambda_e^*\} \in \{(\lambda(\theta_e(t))\boldsymbol{K_e})^{-1}\boldsymbol{u_e}\} \\ \qquad +\boldsymbol{\partial}(I_{\{\widehat{u_e}(t)\}} \circ [\delta_{N_e}])(\{\boldsymbol{u_e}\}) - \{\boldsymbol{f}^*_{\boldsymbol{\theta_e}}(t)\} \quad \text{in } \boldsymbol{V}^*(\{\Omega_e\}) \\ -\{div\ \boldsymbol{u_e}\} \in \{\partial 0_{Y_e}(p_e)\} - \{\widehat{q}_e(t)\} \qquad \text{in } \boldsymbol{Y}^*(\{\Omega_e\}) \\ \text{and } \{\lambda_e^*\} \in \boldsymbol{B}^*(\{\Gamma_e\}) \text{ satisfying the dual synchronizing condition} \\ \{\delta_{\Gamma_e}\boldsymbol{u}_e\} \in \boldsymbol{\partial I}_{Q^*}(\{\lambda_e^*\}) \text{ in } \boldsymbol{B}(\{\Gamma_e\}) \end{cases}$$

## 6 Dual Evolution Macro-Hybrid Variational Formulation

- *Dual evolution macro-hybrid mixed problem*

$$(\boldsymbol{\mathcal{MH}})\begin{cases}\text{Find } \{\boldsymbol{u_{w_e}}\} \in \boldsymbol{\mathcal{V}}_{\{\Omega_e\}} \text{ and } \{\theta_e\} \in \boldsymbol{\mathcal{X}}_{\{\Omega_e\}} \\ \{\boldsymbol{div}^T\theta_e\} - \{\delta^T_{\Gamma_e}\chi^*_e\} \in \{\boldsymbol{\partial F}(\boldsymbol{u_{w_e}})\} - \{h^*_{\theta_e}\} \ \text{ in } \boldsymbol{\mathcal{V}^*}_{\{\Omega_e\}} \\ \{-div\ \boldsymbol{u_{w_e}}\} = \Big\{A_e(\theta_e)\dfrac{\partial\theta_e}{\partial t}\Big\} - \{\hat{q}_{w_e}\} \qquad \text{ in } \boldsymbol{\mathcal{Y}^*}_{\{\Omega_e\}} \\ \{\theta_e(0)\} = \{\hat{\theta}_{0_e}\} \\ \text{and } \{\chi^*_e\} \in \boldsymbol{\mathcal{B}^*}_{\{\Gamma_e\}} \text{ satisfying the dual synchronizing condition} \\ \{\delta_{\Gamma_e}\boldsymbol{u_{w_e}}\} \in \boldsymbol{\partial I}_{Q^*}(\{\chi^*_e\}) \text{ in } \boldsymbol{\mathcal{B}}_{\{\Gamma_e\}}\end{cases}$$

[Alboin 2000]

- *Specific fractured media decomposition*

$$\{\Omega_e\}_{e=1}^{E_R} \ \text{ and } \ \{\Omega_e\}_{e=E_R+1}^{E} \quad E = E_R + E_F \tag{14}$$

- *Macro-scaling asymptotic process*

fracture-system $\{\Omega_f\}_{f=E_R+1}^{E} \longrightarrow$ fracture interface-system $\{\Gamma_f\}_{f=E_R+1}^{E}$

- *Interaction surfaces*

$$\Sigma_{ef} = \partial\Omega_e \cap \Gamma_f \quad 1 \leq e \leq E_R, \quad E_R + 1 \leq f \leq E \tag{15}$$

- *Fracture interface-system pressure contributions*

$$-\Big\{ \sum_{f=E_R+1}^{E} \delta_{\Sigma_{ef}}^T \gamma_{\Sigma_{ef}} p_f \Big\} \ \text{ and } \ -\Big\{ \sum_{f=E_R+1}^{E} \delta_{\Sigma_{ef}}^T \gamma_{\Sigma_{ef}} \theta_f \Big\} \tag{16}$$

- *Rock-system source contributions*

$$-\Big\{ \frac{1}{d} \sum_{e=1}^{E_R} \gamma_{\Sigma_{ef}}^T \delta_{\Sigma_{ef}} \boldsymbol{u}_e \Big\} \ \text{ and } \ -\Big\{ \frac{1}{d} \sum_{e=1}^{E_R} \gamma_{\Sigma_{ef}}^T \delta_{\Sigma_{ef}} \boldsymbol{u}_{w,e} \Big\} \tag{17}$$

# 7 Macroscopic Fractional Fow Problem for the Rock-System

- *Fracture interface-system pressure contribution*

$$-\Big\{ \sum_{f=E_R+1}^{E} \delta^T_{\Sigma_{ef}} \gamma_{\Sigma_{ef}} p_f \Big\} \tag{18}$$

- *Instantaneous total velocity-global pressure-interface global pressure macro-hybrid dual mixed problem*

$$(\boldsymbol{M\widetilde{H}_R}) \begin{cases} \text{Find } \{\boldsymbol{u_e}\} \in \boldsymbol{V}(\{\Omega_e\}) \text{ and } \{p_e\} \in \boldsymbol{Y}(\{\Omega_e\}) \\ \{\boldsymbol{div}^T p_e\} - \{\delta^T_{\Gamma_e} \lambda^*_e\} \in \{(\lambda(\theta_e(t))\boldsymbol{K_e})^{-1}\boldsymbol{u_e}\} + \boldsymbol{\partial}(I_{\{\widehat{u_e}(t)\}} \circ [\delta_e])(\{\boldsymbol{u_e} \\ \qquad -\{\boldsymbol{f^*_{\theta_e}}(t)\} - \Big\{ \sum_{f=E_R+1}^{E} \delta^T_{\Sigma_{ef}} \gamma_{\Sigma_{ef}} p_f \Big\} \qquad \text{in } \boldsymbol{V}^*(\{\Omega_e \\ -\{div\ \boldsymbol{u_e}\} \in \{\partial 0_{Y_e}(p_e)\} - \{\widehat{q}_e(t)\} \qquad \text{in } \boldsymbol{Y}^*(\{\Omega_e \\ \text{and } \{\lambda^*_e\} \in \boldsymbol{B}^*(\{\Gamma_e\}) \text{ satisfying the dual synchronizing condition} \\ \{\delta_{\Gamma_e}\boldsymbol{u}_e\} \in \boldsymbol{\partial I}_{Q^*}(\{\lambda^*_e\}) \quad \text{in } \boldsymbol{B}(\{\Gamma_e\}) \end{cases}$$

- *Fracture interface-system pressure contribution*

$$-\Big\{\sum_{f=E_R+1}^{E}\delta^T_{\Sigma_{ef}}\gamma_{\Sigma_{ef}}\theta_f\Big\} \tag{19}$$

- *Evolution wetting velocity-complementary pressure-interface complementary pressure macro-hybrid dual mixed problem*

$$(\widetilde{\boldsymbol{\mathcal{MH}}_{\boldsymbol{\mathcal{R}}}})\begin{cases}\text{Find } \{\boldsymbol{u_{w_e}}\} \in \boldsymbol{\mathcal{V}}_{\{\Omega_e\}} \text{and } \{\theta_e\} \in \boldsymbol{\mathcal{X}}_{\{\Omega_e\}} \\ \{\boldsymbol{div}^T\theta_e\} - \{\delta^T_{\Gamma_e}\chi^*_e\} \in \{\boldsymbol{\partial F}(\boldsymbol{u_{w_e}})\} - \{\boldsymbol{h}^*_{\boldsymbol{\theta_e}}\} \\ \qquad -\Big\{\sum_{f=E_R+1}^{E}\delta^T_{\Sigma_{ef}}\gamma_{\Sigma_{ef}}\theta_f\Big\} \qquad\qquad \text{in } \boldsymbol{\mathcal{V}}^*{}_{\{\Omega_e\}} \\ \{-div\ \boldsymbol{u_{w_e}}\} = \Big\{A_e(\theta_e)\dfrac{\partial\theta_e}{\partial t}\Big\} - \{\hat{q}_{w_e}\}, \qquad \text{in } \boldsymbol{\mathcal{Y}}^*{}_{\{\Omega_e\}} \\ \{\theta_e(0)\} = \{\hat{\theta}_{0_e}\} \\ \text{and } \{\chi^*_e\} \in \boldsymbol{\mathcal{B}}^*{}_{\{\Gamma_e\}} \text{ satisfying the dual synchronizing condition} \\ \{\delta_{\Gamma_e}\boldsymbol{u_{w_e}}\} \in \boldsymbol{\partial I}_{Q^*}(\{\chi^*_e\}), \quad \text{in } \boldsymbol{\mathcal{B}}_{\{\Gamma_e\}}\end{cases}$$

# 8 Macroscopic Fractional Fow Problem for the Fracture Interface-System

- *Fracture interface internal boundaries and interfaces*

$$\begin{array}{rl} \mathcal{I}_f & f = E_R + 1, ..., E \\ \mathcal{I}_{fg} = \mathcal{I}_f \cap \mathcal{I}_g & E_R + 1 \le f < g \le E \end{array} \tag{20}$$

- *Rock-system source contribution*

$$-\Big\{\frac{1}{d}\sum_{e=1}^{E_R} \gamma^T_{\Sigma_{ef}} \delta_{\Sigma_{ef}} \boldsymbol{u}_e\Big\} \tag{21}$$

- *Instantaneous total velocity-global pressure-interface global pressure macro-hybrid dual mixed problem*

$$(\boldsymbol{M\widetilde{H}_F}) \left\{ \begin{array}{ll} \text{Find } \{\boldsymbol{u_e}\} \in \boldsymbol{V}(\{\Gamma_e\}) \text{ and } \{p_e\} \in \boldsymbol{Y}(\{\Gamma_e\}) & \\ \{\boldsymbol{div}^T p_e\} - \{\delta^T_{\Gamma_e} \lambda^*_e\} \in \{(\lambda(\theta_e(t))\boldsymbol{K_e})^{-1}\boldsymbol{u_e}\} & \\ \quad +\boldsymbol{\partial}(I_{\{\widehat{u_e}(t)\}} \circ [\delta_e])(\{\boldsymbol{u_e}\}) - \{\boldsymbol{f^*_{\theta_e}}(t)\} & \text{in } \boldsymbol{V}^*(\{\Gamma_e\}) \\ -\{div\ \boldsymbol{u_e}\} \in \{\partial 0_{Y_e}(p_e)\} - \{\hat{q}_e(t)\} - \Big\{\frac{1}{d}\sum_{e=1}^{E_R} \gamma^T_{\Sigma_{ef}} \delta_{\Sigma_{ef}} \boldsymbol{u}_e\Big\} & \text{in } \boldsymbol{Y}^*(\{\Gamma_e\}) \\ \text{and } \{\lambda^*_e\} \in \boldsymbol{B}^*(\{\mathcal{I}_e\}) \text{ satisfying the dual synchronizing condition} & \\ \{\delta_{\Gamma_e}\boldsymbol{u}_e\} \in \boldsymbol{\partial I}_{Q^*}(\{\lambda^*_e\}) \quad \text{in } \boldsymbol{B}(\{\mathcal{I}_e\}) & \end{array} \right.$$

- *Rock-system source contribution*

$$-\Big\{\frac{1}{d}\sum_{e=1}^{E_R}\gamma^T_{\Sigma_{ef}}\delta_{\Sigma_{ef}}\boldsymbol{u}_{w,e}\Big\} \tag{22}$$

- *Evolution wetting velocity-complementary pressure-interface complementary pressure macro-hybrid dual mixed problem*

$$(\widetilde{\boldsymbol{\mathcal{MH}}_{\boldsymbol{\mathcal{F}}}})\begin{cases}\text{Find } \{\boldsymbol{u}_{\boldsymbol{w_e}}\} \in \boldsymbol{\mathcal{V}}_{\{\Gamma_e\}} \text{ and } \{\theta_e\} \in \boldsymbol{\mathcal{X}}_{\{\Gamma_e\}} \\ \{\boldsymbol{div}^T\theta_e\} - \{\delta^T_{\Gamma_e}\chi^*_e\} \in \{\boldsymbol{\partial F}(\boldsymbol{u}_{\boldsymbol{w_e}})\} - \{\boldsymbol{h}^*_{\boldsymbol{\theta_e}}\} \quad \text{in } \boldsymbol{\mathcal{V}}^*_{\{\Gamma_e\}} \\ \{-div\ \boldsymbol{u}_{\boldsymbol{w_e}}\} = \Big\{A_e(\theta_e)\dfrac{\partial\theta_e}{\partial t}\Big\} - \{\hat{q}_{w_e}\} - \Big\{\dfrac{1}{d}\sum_{e=1}^{E_R}\gamma^T_{\Sigma_{ef}}\delta_{\Sigma_{ef}}\boldsymbol{u}_{w,e}\Big\} \text{ in } \boldsymbol{\mathcal{Y}}^*_{\{\Gamma_e\}} \\ \{\theta_e(0)\} = \{\hat{\theta}_{0_e}\} \\ \text{and } \{\chi^*_e\} \in \boldsymbol{\mathcal{B}^*}_{\{\mathcal{I}_e\}} \text{ satisfying the dual synchronizing condition} \\ \{\delta_{\Gamma_e}\boldsymbol{u}_{\boldsymbol{w_e}}\} \in \boldsymbol{\partial I}_{Q^*}(\{\chi^*_e\}) \quad \text{in } \boldsymbol{\mathcal{B}}_{\{\mathcal{I}_e\}}\end{cases}$$

# 9 Two-Field Instantaneous Algorithms

- *Models* $(\boldsymbol{MH})$ *and* $(\widetilde{\boldsymbol{MH}}_{\boldsymbol{R}})$-$(\widetilde{\boldsymbol{MH}}_{\boldsymbol{F}})$ *expressed in a classical mixed subdifferential form*

$$
(\mathbf{S})\begin{cases}
\text{Find } \boldsymbol{u} \in \boldsymbol{\mathcal{D}}(\boldsymbol{\mathcal{A}}) \subset \boldsymbol{V} \text{ and } \boldsymbol{p}^* \in \boldsymbol{\mathcal{D}}(G^*) \subset \boldsymbol{Y}^* \\
-\boldsymbol{\Lambda}^T \boldsymbol{p}^* \in \boldsymbol{\mathcal{A}}(\boldsymbol{u}) \quad \text{in } \boldsymbol{V}^* \\
\boldsymbol{\Lambda u} \in \boldsymbol{\partial G}^*(\boldsymbol{p}^*) \quad \text{in } \boldsymbol{Y}
\end{cases}
$$

- *Proximation augmented interpretation*

$$
(\mathbf{S_r})\begin{cases}
\text{Find } \boldsymbol{u} \in \boldsymbol{\mathcal{D}}(\boldsymbol{\mathcal{A}}) \subset \boldsymbol{V} \text{ and } \boldsymbol{p}^* \in \boldsymbol{\mathcal{D}}(G^*) \subset \boldsymbol{Y}^* \\
-\boldsymbol{\Lambda}^T(\boldsymbol{p}^* - \boldsymbol{M}^{-*}\boldsymbol{Prox}_{M^{-*},rG\circ(1/r)I_Y}(\boldsymbol{M}^*\boldsymbol{p}^* + r\boldsymbol{\Lambda u})) \\
\qquad \in (\boldsymbol{\mathcal{A}} + r\boldsymbol{\Lambda}^T\boldsymbol{M}^{-*}\boldsymbol{\Lambda})(\boldsymbol{u}) \qquad \text{in } \boldsymbol{V}^* \\
\boldsymbol{p}^* = (\boldsymbol{M}^{-*} - \boldsymbol{M}^{-*}\boldsymbol{Prox}_{M^{-*},rG\circ(1/r)I_Y})(\boldsymbol{M}^*\boldsymbol{p}^* + r\boldsymbol{\Lambda u}) \quad \text{in } \boldsymbol{Y}^*
\end{cases}
$$

- *Uzawa type algorithm*

**Algorithm I** Given $\boldsymbol{u}^0 \in \boldsymbol{\mathcal{D}}(\boldsymbol{\mathcal{A}}),\ \boldsymbol{p}_0^* \in \boldsymbol{\mathcal{D}}(G^*),$ known $\boldsymbol{u}^m,\ \boldsymbol{p}_m^*,\ m \geq 0$ find $\boldsymbol{u}^{m+1}$ and $\boldsymbol{p}_{m+1}^*$

$$-\boldsymbol{\Lambda}^T(\boldsymbol{p}_m^* - \boldsymbol{M}^{-*}\boldsymbol{Prox}_{M^{-*},rG\circ(1/r)I_Y}(\boldsymbol{M}^*\boldsymbol{p}_m^* + r\boldsymbol{\Lambda}\boldsymbol{u}^m))$$

$$\in (\boldsymbol{\mathcal{A}} + r\boldsymbol{\Lambda}^T\boldsymbol{M}^{-*}\boldsymbol{\Lambda})(\boldsymbol{u}^{m+1}) \quad \text{in } \boldsymbol{V}^*$$

$$\boldsymbol{p}_{m+1}^* = (\boldsymbol{M}^{-*} - \boldsymbol{M}^{-*}\boldsymbol{Prox}_{M^{-*},rG\circ(1/r)I_Y})(\boldsymbol{M}^*\boldsymbol{p}_m^* + r\boldsymbol{\Lambda}\boldsymbol{u}^{m+1}) \quad \text{in } \boldsymbol{Y}^*$$

**Theorem 5** *Proximal-point* **Algorithm I** *is convergent whenever the dual operator condition*

$(\mathbf{C}_{\mathcal{A}^*_{\boldsymbol{\Lambda}},\partial\mathbf{G}^*})\quad -\boldsymbol{\Lambda}\boldsymbol{\mathcal{A}}^{-1}(-\boldsymbol{\Lambda}^{\boldsymbol{T}}(\cdot)) + \boldsymbol{\partial G}^* : \boldsymbol{Y}^* \to \mathbf{2}^{\boldsymbol{Y}}$ *is maximal monotone*

*holds true*

# 10 Three-Field Instantaneous Algorithms

- *Intermediate third primal field*

$$\boldsymbol{\tau} = \boldsymbol{\Lambda u} \in \boldsymbol{Y} \tag{23}$$

- *Models* $(\boldsymbol{MH})$ *and* $(\widetilde{\boldsymbol{MH}}_{\boldsymbol{R}})$-$(\widetilde{\boldsymbol{MH}}_{\boldsymbol{F}})$ *expressed in an extended classical three-field mixed subdifferential form*

$$(\boldsymbol{\mathcal{S}})\begin{cases} \text{Find } (\boldsymbol{u},\boldsymbol{\tau}) \in \boldsymbol{\mathcal{D}}(\boldsymbol{\mathcal{A}}) \times \boldsymbol{\mathcal{D}}(G) \subset \boldsymbol{V} \times \boldsymbol{Y} \text{ and } \boldsymbol{p}^* \in \boldsymbol{\mathcal{D}}(G^*) \subset \boldsymbol{Y^*} \\ -\boldsymbol{\Lambda}^T \boldsymbol{p}^* \in \boldsymbol{\mathcal{A}}(\boldsymbol{u}) \qquad \text{in } \boldsymbol{V^*} \\ \boldsymbol{p}^* \in \boldsymbol{\partial G}(\boldsymbol{\tau}) \qquad \text{in } \boldsymbol{Y^*} \\ \boldsymbol{\Lambda u} - \boldsymbol{\tau} \in \boldsymbol{\partial 0}_{\boldsymbol{Y^*}}(\boldsymbol{p}^*) \quad \text{in } \boldsymbol{Y} \end{cases}$$

- *Proximation augmented interpretation*

$$
(\boldsymbol{\mathcal{S}}_r)\begin{cases}
\text{Find } (\boldsymbol{u},\boldsymbol{\tau}) \in \boldsymbol{\mathcal{D}}(\boldsymbol{\mathcal{A}}) \times \boldsymbol{\mathcal{D}}(G) \subset \boldsymbol{V} \times \boldsymbol{Y} \text{ and } \boldsymbol{p}^* \in \boldsymbol{\mathcal{D}}(G^*) \subset \boldsymbol{Y}^* \\
-\boldsymbol{\Lambda}^T(\boldsymbol{p}^* - r\boldsymbol{M}^{-*}\boldsymbol{\tau}) \in (\boldsymbol{\mathcal{A}} + r\boldsymbol{\Lambda}^T\boldsymbol{M}^{-*}\boldsymbol{\Lambda})(\boldsymbol{u}) \quad \text{in } \boldsymbol{V}^* \\
\boldsymbol{p}^* + r\boldsymbol{M}^{-*}\boldsymbol{\Lambda}\boldsymbol{u} \in \boldsymbol{\partial G}(\boldsymbol{\tau}) + r\boldsymbol{M}^{-*}\boldsymbol{\tau} \quad \text{in } \boldsymbol{Y}^* \\
\boldsymbol{p}^* = \boldsymbol{p}^* + r\boldsymbol{M}^{-*}(\boldsymbol{\Lambda}\boldsymbol{u} - \boldsymbol{\tau}) \quad \text{in } \boldsymbol{Y}^*
\end{cases}
$$

- *Uzawa type algorithm*

**Algorithm II** Given $\boldsymbol{p}_0^* \in \boldsymbol{\mathcal{D}}(G^*)$, known $\boldsymbol{p}_m^*$, find $\boldsymbol{u}^{m+1}$, $\boldsymbol{\tau}^{m+1}$ and $\boldsymbol{p}_{m+1}^*$

$$-\boldsymbol{\Lambda}^T\left(\boldsymbol{p}_m^* - r\boldsymbol{M}^{-*}\boldsymbol{\tau}^{m+1}\right) \in \left(\boldsymbol{\mathcal{A}} + r\boldsymbol{\Lambda}^T\boldsymbol{M}^{-*}\boldsymbol{\Lambda}\right)(\boldsymbol{u}^{m+1}) \quad \text{in } \boldsymbol{V}^*$$

$$\boldsymbol{p}_m^* + r\boldsymbol{M}^{-*}\Lambda\boldsymbol{u}^{m+1} \in \boldsymbol{\partial G}(\boldsymbol{\tau}^{m+1}) + r\boldsymbol{M}^{-*}\boldsymbol{\tau}^{m+1} \quad \text{in } \boldsymbol{Y}^*$$

$$\boldsymbol{p}_{m+1}^* = \boldsymbol{p}_m^* + r\boldsymbol{M}^{-*}(\boldsymbol{\Lambda}\boldsymbol{u}^{m+1} - \boldsymbol{\tau}^{m+1}) \quad \text{in } \boldsymbol{Y}^*$$

**Theorem 6** *Proximal-point* **Algorithm II** *is convergent whenever the dual operator condition*

$(\mathbf{C}_{\mathcal{A}^*_{\boldsymbol{\Lambda}},\partial\mathbf{G}^*})$ $-\boldsymbol{\Lambda}\boldsymbol{\mathcal{A}}^{-1}(-\boldsymbol{\Lambda}^{\boldsymbol{T}}(\cdot)) + \boldsymbol{\partial G}^* : \boldsymbol{Y}^* \to \mathbf{2}^{\boldsymbol{Y}}$ *is maximal monotone is fulfilled*

- *For mesoscopic ($\boldsymbol{\mathcal{MH}}$) and, similarly, macroscopic dual evolution macro-hybrid mixed problems ($\boldsymbol{\mathcal{MH}_R}$)-($\boldsymbol{\mathcal{MH}_F}$)*

## 11 The Operator Splitting Douglas-Rachford Scheme

- *Parallel proximal realization with intermediate hybrid vector*

$$\{\kappa_e^{m+1}\} = \{\delta_{\Gamma_e}\boldsymbol{u}_e^{m+1}\} \in \boldsymbol{\partial I_{Q^*}}(\{\overline{\chi_e^{*m+1}}\}) \subset \boldsymbol{\mathcal{B}}_{\{\Gamma_e\}} \tag{24}$$

**Algorithm $\mathbf{I}_{\mathcal{MH}}$** Given $\{\boldsymbol{u}^{\mathbf{0}}_{\boldsymbol{w_e}}\} \in \boldsymbol{\mathcal{V}}_{\{\Omega_e\}},\ \{\theta_e^0\} \in \boldsymbol{Y}(\{\Omega_e\}),\ \{\chi_e^{*0}\} \in \boldsymbol{Q}^*$
known $\{u_e^m\},\ \{\theta_e^m\},\ \{\chi_e^{*m}\},\ m \geq 0$
find $\{\kappa_e^{m+1}\}$ satisfying the primal synchronizing condition

$$\{\kappa_e^{m+1}\} = \boldsymbol{Proj}_Q\left(\left\{\frac{1}{r}A_e(\theta_e^m)\chi_e^{*m} + \delta_{\Gamma_e}u_e^m\right\}\right)$$

and, in parallel, $u_e^{m+1},\ \theta_e^{m+1},\ \chi_e^{*m+1},\ e = 1, 2, ..., E$

$$\boldsymbol{div}^T\theta_e^m - \delta_{\Gamma_e}^T(\chi_e^{*m} - r\kappa_e^{m+1})$$

$$\in (\boldsymbol{\partial F} + r div^T A_e^{-1}(\theta_e^m) div + r\delta_{\Gamma_e}^T\delta_{\Gamma_e})(\boldsymbol{u}^{\boldsymbol{m}}_{\boldsymbol{w_e}}) - \boldsymbol{h}^{\boldsymbol{*m}}_{\boldsymbol{\theta_e}} - div^T M_e^{-*}\hat{q}_{w_e}^{m+1}$$

$$\theta_e^{m+1} = \theta_e^m + rA_e^{-1}(\theta_e^m)(-div\ u_e^{m+1} + \hat{q}_{w_e}^{m+1})$$

$$\chi_e^{*m+1} = \chi_e^{*m} + r(\delta_{\Gamma_e}u_e^{m+1} - \kappa_e^{m+1})$$

# 12 The Operator Splitting Peaceman-Rachford Scheme

- *Parallel proximal realization with intermediate hybrid vector*

$$\{\kappa_e^{m+1}\} = \{\delta_{\Gamma_e}\boldsymbol{u}_e^m\} \in \boldsymbol{\partial I_{Q^*}}(\{\chi_e^{*m+1/2}\}) \subset \boldsymbol{\mathcal{B}}_{\{\Gamma_e\}} \qquad (25)$$

**Algorithm II**$_{\mathcal{MH}}$ Given $\{\boldsymbol{u^0_{w_e}}\} \in \boldsymbol{\mathcal{V}}_{\{\Omega_e\}},\ \{\theta_e^0\} \in \boldsymbol{Y}(\{\Omega_e\}),\ \{\chi_e^{*0}\} \in \boldsymbol{Q}^*$
known $\{u_e^m\},\ \{\theta_e^m\},\ \{\chi_e^{*m}\},\ m \geq 0$
find $\{\kappa_e^{m+1}\}$ satisfying the primal synchronizing condition

$$\{\kappa_e^{m+1}\} = \boldsymbol{Proj}_Q\left(\left\{\frac{1}{r}A_e(\theta_e^m)\chi_e^{*m} + \delta_{\Gamma_e}u_e^m\right\}\right)$$

and, in parallel, $\theta_e^{*m+1},\ \chi_e^{*m+1},\ u_e^{m+1},\ \theta_e^{m+1},\ \chi_e^{*m+1},\ e = 1, 2, ..., E$

$$\theta_e^{*m+1} = \theta_e^{*m} + r/2A_e^{-1}(\theta_e^m)(-div\ u_e^m + \hat{q}_{w_e}^m)$$

$$\chi_e^{*m+1} = \chi_e^{*m} + r/2\ (\delta_{\Gamma_e}u_e^m - \kappa_e^{m+1})$$

$$\boldsymbol{div}^T\theta_e^{m+1/2} - \delta_{\Gamma_e}^T(\chi_e^{*m+1/2} - r\kappa_e^{m+1})$$

$$\in (\boldsymbol{\partial F} + r div^T A_e^{-1}(\theta_e^m)div + r/2\ \delta_{\Gamma_e}^T\delta_{\Gamma_e})(\boldsymbol{u_{w_e}^{m+1}}) - \boldsymbol{h_{\theta_e}^{*m+1}} - r/2 div^T A_e^{-1}(\theta_e^m$$

$$\theta_e^{*m+1} = \theta_e^{*m+1/2} + r/2A_e^{-1}(\theta_e^m)(-div\ u_e^{m+1} + \hat{q}_{w_e}^{m+1})$$

$$\chi_e^{*m+1} = \chi_e^{*m+1/2} + r/2\ (\delta_{\Gamma_e}u_e^{m+1} - \kappa_e^{m+1})$$

# 13 Convergence of the Operator Splitting Schemes

- *Mosco's dual operator of* $\boldsymbol{A}_{\boldsymbol{h}_{\boldsymbol{\theta}}^*} = [\partial F_e](\cdot) - \{h_{\theta_e}^*\}$ *relative to coupling operator* $\boldsymbol{D} = ([-div], [\delta_{\Gamma_e}])$

$$\boldsymbol{A}^*_{h^*_\theta,D}(\boldsymbol{\mu},\boldsymbol{\nu}^*) \equiv (\boldsymbol{A}^*_{h^*_\theta,-div}(\boldsymbol{\mu}), \boldsymbol{A}^*_{h^*_\theta,\delta_{\Gamma_e}}(\boldsymbol{\nu}^*) = \{(\boldsymbol{\xi},\boldsymbol{\zeta}^*) \in \boldsymbol{\mathcal{Y}}_{\{\Omega_e\}} \times \boldsymbol{\mathcal{B}^*}_{\{\Gamma_e\}} :$$

$$\exists \boldsymbol{\beta} \in \boldsymbol{\mathcal{V}}_{\{\Omega_e\}},\ \boldsymbol{\xi}^* = [div]\boldsymbol{\beta},\ \boldsymbol{\zeta}^* = -[\delta_{\Gamma_e}]\boldsymbol{\beta}, -[div^T]\boldsymbol{\mu}^* - [\delta^T_{\Gamma_e}]\boldsymbol{\nu}^* \in \boldsymbol{A}_{h^*_\theta}(\boldsymbol{\beta})\}$$

- *Auxiliary dual supervector* $\boldsymbol{s}^* \in \boldsymbol{\mathcal{Y}}_{\{\Omega_e\}} \times \boldsymbol{\mathcal{B}^*}_{\{\Gamma_e\}}$

$$\boldsymbol{\mathcal{M}}^*\boldsymbol{s}^* \in (\boldsymbol{\mathcal{M}}^* + r\boldsymbol{A}^*_{h^*_\theta,D})(\{\theta_e\},\{\chi^*_e\}) \iff (\{\theta_e\},\{\chi^*_e\}) = \boldsymbol{J}_{\mathcal{M}^*,A^*_{h^*_\theta,D}}(\boldsymbol{\mathcal{M}}^*\boldsymbol{s}^*)$$

- *Douglas-Rachford macro-hybrid dual problem at* $m+1 \geq 1$ *time step*

$$(\boldsymbol{D}^{m+1}) \begin{cases} \text{Given } \{\theta^0_e = \widehat{\theta}_{0_e} \in Z(\Omega_e)\},\ \{\chi^{*0}_e\} \in \boldsymbol{\mathcal{Q}^*},\ \boldsymbol{s}^{*-1} \in \boldsymbol{\mathcal{Y}}_{\{\Omega_e\}} \times \boldsymbol{\mathcal{B}^*}_{\{\Gamma_e\}} \\ \text{find } \{\theta^{m+1}_e\},\ \{\chi^{*m+1}_e\} \text{ and } \{s^m_e\} \\ \\ \boldsymbol{\mathcal{M}}^*\boldsymbol{s}^{*m} \in (\boldsymbol{\mathcal{M}}^* + r\boldsymbol{A}^*_{h^*_\theta,D})(\{\theta^{*m+1}_e\},\{\chi^{*m+1}_e\}) \\ \\ 2\boldsymbol{\mathcal{M}}^*(\{\theta^{*m}_e\},\{\chi^{*m}_e\}) - \boldsymbol{\mathcal{M}}^*\boldsymbol{s}^{*m-1} \\ \\ \qquad \in (\boldsymbol{\mathcal{M}}^* + r\boldsymbol{\partial\mathcal{G}}^*)((\{\theta^{*m}_e\},\{\chi^{*m}_e\}) + \boldsymbol{s}^{*m} - \boldsymbol{s}^{*m-1}) \end{cases}$$

• *Peaceman-Rachford macro-hybrid dual problem at* $m+1 \geq 1$

$$
(\widetilde{\boldsymbol{D}^{m+1}}) \begin{cases} \text{Given } \{\theta_e^0 = \hat{\theta}_{0_e} \in Z(\Omega_e)\},\ \{\chi_e^{*0}\} \in \boldsymbol{\mathcal{Q}}^*,\ \boldsymbol{s}^{*-1} \in \boldsymbol{\mathcal{Y}}_{\{\Omega_e\}} \times \boldsymbol{\mathcal{B}}^*{}_{\{\Gamma_e\}} \\ \text{find } \{\theta_e^{m+1}\},\ \{\chi_e^{*m+1}\} \text{ and } \{s_e^m\} \\ \\ \boldsymbol{\mathcal{M}}^* \boldsymbol{s}^{*m} \in (\boldsymbol{\mathcal{M}}^* + r/2\ \boldsymbol{A}^*_{h^*_\theta, D})(\{\theta_e^{*m+1}\}, \{\chi_e^{*m+1}\}) \\ \\ 2\boldsymbol{\mathcal{M}}^*(\{\theta_e^{*m}\}, \{\chi_e^{*m}\}) - \boldsymbol{\mathcal{M}}^* \boldsymbol{s}^{*m-1} \\ \\ \qquad \in (\boldsymbol{\mathcal{M}}^* + r\boldsymbol{\partial}\boldsymbol{\mathcal{G}}^*)((\{\theta_e^{*m}\}, \{\chi_e^{*m}\}) + r/2\ \boldsymbol{s}^{*m} - r/2\ \boldsymbol{s}^{*m-1}) \end{cases}
$$

**Theorem 6** *Let dual operators* $\boldsymbol{A}^*_{h^*_\theta, -div}$ *and* $\boldsymbol{A}^*_{h^*_\theta, \delta_{\Gamma_e}}$ *be maximal monotone. Then, for time-independent data* $\boldsymbol{f}^{*m} = \boldsymbol{f}^*$, $\boldsymbol{w}_g^{m+1} = \boldsymbol{w}_g$, *operator splitting algorithms* **Algorithm I**$_{\widetilde{\mathcal{MH}}}$ *and* **Algorithm II**$_{\mathcal{MH}}$ *evolve, as* $m \to \infty$, *to a* $\boldsymbol{D}^{m+1}$*- and a* $\widetilde{\boldsymbol{D}^{m+1}}$*-stationary state of the dual evolution macro-hybrid mixed problem* ($\boldsymbol{\mathcal{MH}}$)*, respectively*

# Conclusions

- *Subdifferential variational modeling in subsurface flow: theory à la Moreau-Duvaut-Lions-Temam-Le Tallec*
- *Coupling surjectivity compatibility for compositional dualization*
- *Dual stationary and evolution solvability duality principles*
- *Variational macro-hybridization for scaling, localization, multi-constitutivity, multi-algorithmia and parallel computing*
- *Proximation augmented penalty-duality stationary algorithms*
- *Proximation realization of semi-implicit time marching schemes*
- *Variational basis for semi-discrete and fully discrete approximations*