Domain Decomposition method for Reaction-Diffusion Systems

Rodrigue Kammogne[1] and Daniel Loghin[1]

1 Introduction

Reaction diffusion systems have important applications in the area of modern mathematical modeling. They can be found in a number of real-life problems, ranging from chemical and biological phenomena to medicine, for example [5, 10]. However the numerical solution to reaction-diffusion problems remains a challenge, as they are often represented as a system of nonlinear PDEs, which are solved on a complex domain. One approach to attempt to solve such problems is to use domain decomposition methods (**DD**), which are more powerful and flexible. They deal with the problem in a more elegant and efficient way, by dividing the domain into subdomains and then obtaining the solution by solving smaller problems on these subdomains.
In a recent paper, Caetano et al. [3] have introduced a non-overlapping domain decomposition algorithm of Schwarz waveform relaxation type for semilinear reaction-diffusion equations. For solving the interface problem they proposed a new type of nonlinear transmission, using Robin or Ventcell transmission conditions, which leads to a solution technique independent of the mesh parameter. However, this has not been extended to reaction-diffusion systems. Our aim in this work is to present an alternative approach to approximate the Steklov-Poincaré operators arising from a non-overlapping **DD**-algorithm for reaction diffusion systems. Our approach is related to that in [2]. The coercivity and the continuity of the Steklov-Poincaré operators arising in a non-overlapping domain decomposition algorithm for scalar elliptic problems with respect to Sobolev norms of index $1/2$ allow us to construct a new interface preconditioner, which leads to solution techniques independent of the mesh size h. We validate the theoretical results on various numerical experiments.

2 Problem Description

Let $\Omega \subset \mathbb{R}^2$ be an open bounded set. We consider the following model problem:

$$\begin{cases} -\mathrm{D}\Delta\mathbf{u} + \mathrm{M}\mathbf{u} = \mathbf{f} & \text{in } \Omega, \\ \mathbf{u} = \mathbf{0} & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where:

$$\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \quad \mathrm{M} = \begin{pmatrix} \alpha_1(x,y) & \beta_1(x,y) \\ \beta_2(x,y) & \alpha_2(x,y) \end{pmatrix}, \quad \mathbf{f} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \quad \mathrm{D} = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}.$$

[1] University of Birmingham, Edgbaston, Birmingham B15 2TT, United Kingdom,
e-mail: kammognr@for.mat.bham.ac.uk, e-mail: d.loghin@bham.ac.uk

We assume that f_1 and f_2 are in $L^2(\Omega)$ and M satisfies the following bounds for all $(x,y) \in \Omega$:

$$0 < \gamma_{min} < \frac{\xi^T \mathrm{M} \xi}{\xi^T \xi} \quad \text{for all} \quad \xi \in \mathbb{R}^2 \setminus \{0\} \quad \text{and} \quad \|M\| < \gamma_{max}. \tag{2}$$

The weak formulations of problem (1) reads:

$$\begin{cases} \text{Find } \mathbf{u} \in H_0^1(\Omega) \times H_0^1(\Omega) \text{ such that for all } \mathbf{z} \in H_0^1(\Omega) \times H_0^1(\Omega) \\ B(\mathbf{u},\mathbf{z}) = <\mathbf{f},\mathbf{z}>, \end{cases} \tag{3}$$

where:

$$B(\mathbf{w},\mathbf{z}) = \int_\Omega \mathrm{D}\nabla\mathbf{w} : \nabla\mathbf{z} + (\mathrm{M}\mathbf{w}) \cdot \mathbf{z}\, dx, \text{ and } <\mathbf{f},\mathbf{z}> = \int_\Omega \mathbf{f} \cdot \mathbf{z}\, dx.$$

For the weak form (3), it can be shown that the conditions of the Lax-Milgram lemma are satisfied (see [4] for more details). In particular,

$$B(\mathbf{u},\mathbf{z}) \leq \max\{1,\gamma_{max}\}\|\mathbf{u}\|_1\|\mathbf{z}\|_1, \quad \forall \mathbf{u},\mathbf{z} \in H_0^1(\Omega) \times H_0^1(\Omega), \tag{4}$$

$$B(\mathbf{z},\mathbf{z}) \geq \min\{1,\gamma_{min}\}\|\mathbf{z}\|_1^2, \quad \forall \mathbf{z} \in H_0^1(\Omega) \times H_0^1(\Omega) \tag{5}$$

Let $V^h \times V^h$ be a finite dimensional subspace of $H_0^1(\Omega) \times H_0^1(\Omega)$. The finite element discretizations of the weak formulation (3) reads:

$$\begin{cases} \text{Find } \mathbf{u}_h \in V^h \times V^h \text{ such that for all } \mathbf{z}_h \in V^h \times V^h \\ B(\mathbf{u}_h,\mathbf{z}_h) = \langle \mathbf{f}_h, \mathbf{z}_h \rangle. \end{cases} \tag{6}$$

Since (4), (5) hold for all $\mathbf{u}$, $\mathbf{z} \in H_0^1(\Omega) \times H_0^1(\Omega)$, the existence and uniqueness of the solution of formulation (6) is guaranteed by the Lax-Milgram lemma for all $\mathbf{u}_h$, $\mathbf{z}_h \in V^h \times V^h$.

3 Domain decomposition

Let Ω be partitioned into N subdomains without overlap such that:

$$\overline{\Omega} = \bigcup_{i=1}^{N} \overline{\Omega}_i, \quad \Omega_i \cap \Omega_j = \emptyset \ (i \neq j), \quad \Gamma_i = \partial\Omega_i \backslash \partial\Omega, \quad \Gamma = \bigcup_{i=1}^{N} \Gamma_i.$$

Let also $\mathbf{u}_i = \mathbf{u}\,|_{\Omega_i}$ be the restriction of the solution $\mathbf{u}$ to subdomain Ω_i, and $\mathbf{u}_i\,|_{\Gamma_i} = \boldsymbol{\lambda}_i$ the trace of $\mathbf{u}$ on each interface.
Problem (1) is equivalent to a set of N subproblems:

$$\begin{cases} \mathscr{L}\mathbf{u}_i = \mathbf{f} & \text{in } \Omega_i, \\ \mathbf{u}_i = \mathbf{0} & \text{on } \partial\Omega_i \cap \partial\Omega, \\ \mathbf{u}_i = \boldsymbol{\lambda_i} & \text{on } \Gamma_i, \end{cases} \tag{7}$$

where $\mathscr{L} := -\mathrm{D}\Delta + \mathrm{M}$. If we write $\mathbf{u}_i = \mathbf{w}_i + \mathbf{v}_i$, then equations (7) are equivalent to the following two sets of N subproblems:

$$\begin{cases} \mathscr{L}\mathbf{w}_i = \mathbf{f} \text{ in } \Omega_i; \\ \mathbf{w}_i = \mathbf{0} \text{ on } \partial\Omega_i \cap \partial\Omega; \\ \mathbf{w}_i = \mathbf{0} \text{ on } \Gamma_i; \end{cases} \quad (8) \qquad \begin{cases} \mathscr{L}\mathbf{v}_i = \mathbf{0} \text{ in } \Omega_i; \\ \mathbf{v}_i = \mathbf{0} \text{ on } \partial\Omega_i \cap \partial\Omega; \\ \mathbf{v}_i = \boldsymbol{\lambda_i} \text{ on } \Gamma_i. \end{cases} \quad (9)$$

We can view $\mathbf{v}_i$ as the $\mathscr{L}$-extension of $\boldsymbol{\lambda}_i$ to the domain Ω_i and will be denoted by $H_i\boldsymbol{\lambda}_i$. The equation for $\boldsymbol{\lambda}$ can be shown to be of the form:

$$\sum_{i=1}^{N} \int_{\Gamma_i} (\mathbf{n}_i \cdot \nabla H_i\boldsymbol{\lambda}_i) \cdot \mathbf{z}_i \, ds = -\sum_{i=1}^{N} \int_{\Gamma_i} (\mathbf{n}_i \cdot \nabla \mathbf{w}_i) \cdot \mathbf{z}_i \, ds. \quad (10)$$

From (10), the Steklov-Poincaré operator $\mathscr{S}$ can be defined in the following way:

$$\langle \mathscr{S}\boldsymbol{\lambda}, \boldsymbol{\mu} \rangle := \sum_{i=1}^{N} \int_{\Gamma_i} (\mathbf{n}_i \cdot \nabla H_i\boldsymbol{\lambda}_i) \cdot \boldsymbol{\mu}_i \, ds. \quad (11)$$

The systems (8) and (9) together with the Steklov-Poincaré problem (10) represent the multi-domain formulation of the problem (1).

3.1 Mixed finite element discretisation

The weak formulation of the multi-domain formulation of the problem (1) reads:

$$(1) \begin{cases} \text{Find } \mathbf{u}_i \in H_0^1(\Omega_i) \times H_0^1(\Omega_i) \text{ such that for all } \mathbf{z}_i \in H_0^1(\Omega_i) \times H_0^1(\Omega_i); \\ B_i(\mathbf{w}_i, \mathbf{z}_i) = (\mathbf{f}_i, \mathbf{z}_i). \end{cases}$$

$$(2) \begin{cases} \text{Find } \boldsymbol{\lambda} \in H_{00}^{1/2}(\Gamma) \times H_{00}^{1/2}(\Gamma) \text{ such that for all } \boldsymbol{\eta} \in H_{00}^{1/2}(\Gamma) \times H_{00}^{1/2}(\Gamma); \\ s(\boldsymbol{\lambda}, \boldsymbol{\eta}) := \langle \mathscr{S}\boldsymbol{\lambda}, \boldsymbol{\eta} \rangle = \sum_{i=1}^{N} [(\mathbf{f}_i, \boldsymbol{\eta}_i) - B_i(\mathbf{w}_i, \boldsymbol{\eta}_i)]. \end{cases}$$

$$(3) \begin{cases} \text{Find } \widetilde{\mathbf{v}}_i \in H_0^1(\Omega_i) \times H_0^1(\Omega_i) \text{ such that for all } \mathbf{z}_i \in H_0^1(\Omega_i) \times H_0^1(\Omega_i); \\ B_i(\widetilde{\mathbf{v}}_i, \mathbf{z}_i) = B_i(\mathbf{v}_i, \mathbf{z}_i) - B_i(\mathbf{p}_i, \mathbf{z}_i) = -B_i(\mathbf{p}_i, \mathbf{z}_i). \end{cases}$$

Note that $\widetilde{\mathbf{v}}_i = \mathbf{v}_i - \mathbf{p}_i$, where $\mathbf{p}_i$ is an $\mathscr{L}$-extension of $\boldsymbol{\lambda}_i$ to Ω_i satisfying $p_i = 0$ on $\partial\Omega_i \cap \partial\Omega$.

Let $\mathfrak{T}_h$ denote a subdivision of $\Omega \subset \mathbb{R}^2$ into simplices. We define $V^h = \bigcup_{i=1}^{N} V_i^h$ a subset of $H_0^1(\Omega)$ to be a space of piecewise polynomial functions on $\mathfrak{T}_h$ such that:

$$V_i^h = V_i^{h,r} := \Big\{ w \in C^0(\Omega_i) : w|_{\mathfrak{t}} \in P_r \ \ \forall \mathfrak{t} \in \mathfrak{T}_h, \ w\,|_{\partial\Omega \cap \partial\Omega_i} = 0 \Big\}.$$

Here $P_r(\mathfrak{t})$ is considered as the space of polynomials in d variables of degree r defined on a set $\mathfrak{t} \subset \mathbb{R}^d$. Given a basis $\{\boldsymbol{\phi}_k\}_{k=1}^{n}$ of $V^h \times V^h$, such that:

$$\mathbf{u}_h(\mathrm{x}) = \sum_{k}^{2(n_I+n_\Gamma)} u_k \boldsymbol{\phi}_k(\mathrm{x}),$$

we obtain the following linear system:

$$\begin{pmatrix} A_{II}^{(1)} & A_{I\Gamma}^{(1)} & M_{II}^{(1)} & M_{I\Gamma}^{(1)} \\ A_{\Gamma I}^{(1)} & A_{\Gamma\Gamma}^{(1)} & M_{\Gamma I}^{(1)} & M_{\Gamma\Gamma}^{(1)} \\ M_{II}^{(2)} & M_{I\Gamma}^{(2)} & A_{II}^{(2)} & A_{I\Gamma}^{(2)} \\ M_{\Gamma I}^{(2)} & M_{\Gamma\Gamma}^{(2)} & A_{\Gamma I}^{(2)} & A_{\Gamma\Gamma}^{(2)} \end{pmatrix} \begin{pmatrix} u_{1I} \\ u_{1\Gamma} \\ u_{2I} \\ u_{2\Gamma} \end{pmatrix} = \begin{pmatrix} f_{1I} \\ f_{1\Gamma} \\ f_{2I} \\ f_{2\Gamma} \end{pmatrix}; \tag{12}$$

with $A^{(i)} := d_i L + \alpha_i M$ and $M^{(i)} := \beta_i M$. The matrix M is known as the mass matrix, while L represents the discrete Laplacian matrix. We also denote by $S_{A^{(i)}} := A_{\Gamma\Gamma}^{(i)} - A_{\Gamma I}^{(i)}(A_{II}^{(i)})^{-1}A_{I\Gamma}^{(i)}$ the corresponding local Schur complement associated with $A^{(i)}$. Equation (12) can be rewritten as:

$$A\mathbf{u} = \begin{pmatrix} A_{II} & A_{I\Gamma} \\ A_{\Gamma I} & A_{\Gamma\Gamma} \end{pmatrix} \begin{pmatrix} \mathbf{u}_I \\ \mathbf{u}_\Gamma \end{pmatrix} = \begin{pmatrix} \mathbf{f}_I \\ \mathbf{f}_\Gamma \end{pmatrix}, \tag{13}$$

where:

$$A_{\mu\nu} = \begin{pmatrix} d_1 L_{\mu\nu} + \alpha_1 M_{\mu\nu} & \beta_1 M_{\mu\nu} \\ \beta_2 M_{\mu\nu} & d_2 L_{\mu\nu} + \alpha_2 M_{\mu\nu} \end{pmatrix}, \quad \mu,\nu \in \{I,\Gamma\}.$$

4 A blockdiagonal interface preconditioner

Let $H_{00}^{1/2}(\Gamma)$ denote the interpolation space between $H_0^1(\Gamma)$ and $L^2(\Gamma)$, which is equipped with the norm $\|\,.\,\|_{1/2,\Gamma}$ as given in [8, chapter 1]. It can be shown that the finite element matrix representation of the norm $\|\,.\,\|_{1/2,\Gamma}$ is given by [1]

$$H_{1/2} := [M_\Gamma, L_\Gamma]_{1/2} := M_\Gamma (M_\Gamma^{-1} L_\Gamma)^{1/2},$$

where M_Γ and L_Γ represent repectively the Mass matrix and discrete Laplacian matrix assembled on Γ. It has been proven in [6] that the matrix $H_{1/2}^{(i)}(\Gamma)$

$$H_{1/2}^{(i)}(\Gamma) := [M_\Gamma, A_\Gamma^{(i)}]_{1/2} := M_\Gamma (M_\Gamma^{-1} A_\Gamma^{(i)})^{1/2}$$

is spectrally equivalent to $H_{1/2}$ for $i = 1,2$, where $A_\Gamma^{(i)} := d_i L_\Gamma + \alpha_i M_\Gamma$.
Consider the following eigenvalue problem:

$$\begin{pmatrix} A_{II} & A_{I\Gamma} \\ A_{\Gamma I} & A_{\Gamma\Gamma} \end{pmatrix} \begin{pmatrix} \mathbf{u}_I \\ \mathbf{u}_\Gamma \end{pmatrix} = \mu \begin{pmatrix} A_{II} & A_{I\Gamma} \\ 0 & P_S \end{pmatrix} \begin{pmatrix} \mathbf{u}_I \\ \mathbf{u}_\Gamma \end{pmatrix} \tag{14}$$

with $S = A_{\Gamma\Gamma} - A_{\Gamma I}A_{II}^{-1}A_{\Gamma I}$. Then $\mu = 1$ or it satisfies:

$$S\mathbf{u}_\Gamma = \mu P_S \mathbf{u}_\Gamma.$$

Using the definition of $\mathscr{S}$ in equation (11), we can derive the following theorem:

Theorem 1. *There exist positive constants c_1, c_2 such that for all $\boldsymbol{\lambda}_h, \boldsymbol{\mu}_h \in H_{00}^{1/2}(\Gamma) \times H_{00}^{1/2}(\Gamma)$:*

$$c_1 \|\boldsymbol{\lambda}_h\|_{1/2,\Gamma}^2 \leq \langle \mathscr{S}\boldsymbol{\lambda}_h, \boldsymbol{\lambda}_h \rangle, \quad \langle \mathscr{S}\boldsymbol{\lambda}_h, \boldsymbol{\mu}_h \rangle \leq c_2 \|\boldsymbol{\lambda}_h\|_{1/2,\Gamma} \|\boldsymbol{\mu}_h\|_{1/2,\Gamma}.$$

Proof. The reader should refer to [6].

From the equivalence between the continuous and the discrete interpolation norms of index $1/2$, we have:

$$\kappa_1 \|\eta_h\|_{1/2,\Gamma} \leq \|\boldsymbol{\eta}\|_{H_{1/2}} \leq \kappa_2 \|\eta_h\|_{1/2,\Gamma}, \quad \forall \boldsymbol{\eta} \in \mathbb{R}^{n_\Gamma}.$$

Therefore we can derive the following inequalities:

Corollary 1. *There exist positive constants $c_1, c_2, \kappa_1, \kappa_2$ such that for all $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathbb{R}^{n_\Gamma}$:*

$$\frac{c_1}{\kappa_2^2} \|\boldsymbol{\lambda}\|_{H_{1/2}^{(1)} \oplus H_{1/2}^{(2)}}^2 \leq \langle S\boldsymbol{\lambda}, \boldsymbol{\lambda} \rangle, \quad \langle S\boldsymbol{\lambda}, \boldsymbol{\mu} \rangle \leq \frac{c_2}{\kappa_1^2} \|\boldsymbol{\lambda}\|_{H_{1/2}^{(1)} \oplus H_{1/2}^{(2)}} \|\boldsymbol{\mu}\|_{H_{1/2}^{(1)} \oplus H_{1/2}^{(2)}}.$$

This leads to the following two remarks:

Remark 1. It can be shown using a standard GMRES convergence based on the Field of Values that any symmetric positive definite preconditioner P_S which satisfies:

$$\xi_2 \|\boldsymbol{\lambda}\|_{P_S}^2 \leq \langle S\boldsymbol{\lambda}, \boldsymbol{\lambda} \rangle, \quad \langle S\boldsymbol{\lambda}, \boldsymbol{\mu} \rangle \leq \xi_1 \|\boldsymbol{\lambda}\|_{P_S} \|\boldsymbol{\mu}\|_{P_S}, \ \forall \boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathbb{R}^n,$$

leads to convergence independent of the size of the problem [9].

Remark 2. It has been shown in [6], that there exist constants σ_i, δ_i such that for all $\widetilde{\lambda}, \widetilde{\mu} \in H_{00}^{1/2}(\Gamma)$:

$$\sigma_i \|\widetilde{\lambda}\|_{H_{1/2}^{(i)}}^2 \leq \langle S_{A^{(i)}} \widetilde{\lambda}, \widetilde{\lambda} \rangle, \quad \langle S_{A^{(i)}} \widetilde{\lambda}, \widetilde{\mu} \rangle \leq \delta_i \|\widetilde{\lambda}\|_{H_{1/2}^{(i)}} \|\widetilde{\mu}\|_{H_{1/2}^{(i)}}; \quad i = 1, 2.$$

Then, a natural choice for P_S is:

$$\widehat{S}_1 = \begin{pmatrix} S_{A^{(1)}} & 0 \\ 0 & S_{A^{(2)}} \end{pmatrix}.$$

Another more practical choice for P_S is:

$$\widehat{S}_2 = \begin{pmatrix} H_{1/2}^{(1)} & 0 \\ 0 & H_{1/2}^{(2)} \end{pmatrix}.$$

The implementation of this preconditioner can be achieved using sparse linear algebra techniques. In particular the action of the inverse of $H_{1/2}^{(i)}$ on a given vector $\mathbf{z} \in \mathbb{R}^n$ can be approximated via a generalised Lanczos algorithms (see [1, 2]), which would only involve sparse computations with interface mass and Laplacian matrices.

5 Numerical results

In this section we present the numerical experiments obtained by solving some reaction diffusion problems in two dimensions. All the problems are solved on a square domain $\Omega = (-1,1)^2$. The domain Ω is divided into $N = N_x \times N_y$ subdomains of size $2/N_x \times 2/N_y$ each, with $N_x = N_y \in \{2,4,8\}$. Furthermore, we used a uniform triangulation on each subdomain so that we work with a sequence of nested grids as well as nested subdomain partitions. The GMRES method is employed with a tolerance of 10^{-6} together with the following right preconditioners:

$$P_{R_j} = \begin{pmatrix} A_{II} & A_{I\Gamma} \\ 0 & \widehat{S}_j \end{pmatrix} \quad (j = 1,2).$$

5.1 Test problem 1

We consider now the problem (1), with the following parameters:

$$d_1 = d_2 = 1, \alpha_1 = \alpha_2 = 10^{k_1}, \beta_1 = \beta_2 = 1$$

with $\mathbf{f}$ such that $\mathbf{u}^T = ((x - \frac{1}{3}x^3)(y - \frac{1}{3}y^3), (x - \frac{1}{3}x^3)(y - \frac{1}{3}y^3) + 2)$. We showed in Table 1 that P_{R_1} is an optimal preconditioner for problem (1), as the number of iterations is independent of the problem size and the number of subdomains. However, it remains computationally expensive. A more practical option is P_{R_2}. We find indeed that working with P_{R_2} still gives us virtually no dependence on the size of the problem but a dependence on the number of subdomains. However this dependence disappears for increasing α_i. This latter property is due to the fact that the problem becomes *'easier'* to solve iteratively as the mass matrix becomes more and more dominant. For the remaining test problems, we consider only P_{R_2}.

Preconditioner=	P_{R_1}						P_{R_2}					
k_1=	1			2			3			1			2			3		
domains =	4	16	64	4	16	64	4	16	64	4	16	64	4	16	64	4	16	64
size = 8,450	4	4	4	4	4	4	4	4	4	14	16	19	13	13	14	11	11	11
33,282	4	4	4	4	4	4	4	4	4	14	16	20	13	13	15	11	11	12
132,098	4	4	4	4	4	4	4	4	4	14	16	20	13	14	15	11	11	12

Table 1 GMRES iterations for Problem 1.

5.2 Test problem 2

We solve the same problem as in the previous example but with $d_1 = 1, d_2 = 0.1$ and $k_2 = 0$. Since $d_1 \neq d_2$ two set of results have been obtained (see Table 2). The first set of results is obtained by applying the preconditioner directly to the problem (1). The second set of results is obtained by applying the preconditioner to a scaled version of problem (1), namely:

$$-\Delta \mathbf{v} + \mathrm{M}\mathrm{D}^{-1}\mathbf{v} = \mathbf{f}, \quad \text{where } \mathbf{v} = \mathrm{D}\mathbf{u}. \tag{15}$$

In both cases we have a logarithmic dependence on the number of subdomains and virtually no dependence on the size of the problem. However the number of iterations remains higher than those seen in test problem 1. This is due to the fact that the preconditioned matrices are no longer symmetric.

	Without Scaling			With Scaling		
k_1=	1	2	3	1	2	3
domains =	4 16 64	4 16 64	4 16 64	4 16 64	4 16 64	4 16 64
size = 8,450	20 24 26	17 19 19	16 17 16	13 14 17	12 12 13	9 11 10
33,282	20 24 27	17 20 21	15 19 19	13 14 18	12 13 13	10 12 11
132,098	20 24 28	18 21 22	15 19 19	14 14 18	12 13 13	10 12 12

Table 2 GMRES iterations for Problem 2 .

Remark 3. The similarity between the second part of the results in Table 1 and Table 2 tells us that the performance of our preconditioner will not be affected if $d_1 << d_2$. In that case the scaled version (15) of the problem is used .

5.3 Test problem 3

Finally we consider problem (1) with $d_1 = 1; d_2 = 0.1; \mathbf{f} = (1,1)^T$ and $\mathbf{u} = 0$ on $\partial\Omega$ together with the following jump coefficients:

$$\alpha_1 = \begin{cases} 1 & \text{if } x^2+y^2 < 1/4 \\ 100 & \text{otherwise} \end{cases}; \quad \alpha_2 = \begin{cases} 100 & \text{if } x^2+y^2 < 1/4 \\ 1 & \text{otherwise} \end{cases}$$

$$\beta_1 = \begin{cases} 0.1 & \text{if } x^2+y^2 < 1/4 \\ 1 & \text{otherwise} \end{cases}; \quad \beta_2 = \begin{cases} 1 & \text{if } x^2+y^2 < 1/4 \\ 0.1 & \text{otherwise} \end{cases}$$

An illustration of the final solution $\mathbf{u}$ is provided in Figure 1, while the iteration count is presented in Table 3. We observe a similar convergence behavior: independence of the problem size and logarithmic dependence on the number of subdomains.

6 Conclusion

We presented a general non-overlapping domain decomposition method for solving a system of coupled reaction-diffusion equations (linear case only). We derived the

domains =	4	16	64
size = 8,450	19	24	28
33,282	18	25	28
132,098	18	26	28

Table 3 GMRES iterations for Problem 3.

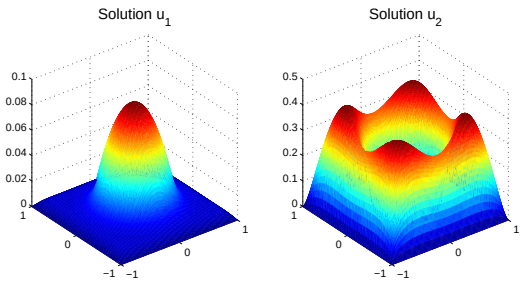

Fig. 1 Solution for Problem 3.

corresponding Steklov-Poincaré operator together with the associated linear algebra problem. In addition, by exploiting the fact that the Steklov-Poincaré operators arising in a non-overlapping **DD**-algorithm are coercive and continuous with respect to Sobolev norms of index $1/2$, an interface preconditioner for the Schur complement problem was constructed, which is strongly related to the finite element representation of the norm $\| \cdot \|_{1/2,\Gamma}$. Its implementation can be achieved via sparse Lanczos procedures, which do not add to the complexity of the problem . We used various numerical examples to validate our theoretical results. We found that the performance of the method is independent of the mesh size h, but remains at worst logarithmically dependent on the number of subdomains. Similar performance is obtained when using a METIS [7] partitioning of the domain, or when our approach is extended to non-linear reaction-diffusion systems (see [6] for more details).

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