Discontinuous Galerkin and Nonconforming in Time Optimized Schwarz Waveform Relaxation for Heterogeneous Problems

Laurence Halpern and Caroline Japhet

LAGA, Université Paris XIII, 99 Avenue J-B Clément, 93430 Villetaneuse, France, {halpern,japhet}@math.univ-paris13.fr

Summary. We consider the question of domain decomposition for evolution problems with discontinuous coefficients. We design a method relying on four ingredients: extension of the optimized Schwarz waveform relaxation algorithms as described in [1], discontinuous Galerkin methods designed in [7], time windows, and a generalization of the projection procedure given in [6]. We so obtain a highly performant method, which retains the approximation properties of the discontinuous Galerkin method. We present numerical results, for a two-domains splitting, to analyze the time-discretization error and to illustrate the efficiency of the DGSWR algorithm with many time windows. This analysis is in continuation with the approach initiated in DD16 [2, 5], with applications in climate modeling, or nuclear waste disposal simulations.

1 Introduction

In order to be able to perform long time computations in highly discontinuous media, it is of importance to split the computation into subproblems for which robust and fast solvers can be used. This happens for instance in climate modeling, where heterogeneous climatic models must be run in parallel, or in nuclear waste disposal simulations, where different materials have different behaviors.

Optimized Schwarz waveform relaxation algorithms have proven to provide an efficient approach for convection-diffusion problems in one [1] and two dimensions [8]. The SWR algorithms are global in time, and therefore are well adapted to coupling models; they lead to fast and efficient solvers, and they allow for the use of non conforming space-time discretizations. Based on this approach, our final objective is to propose efficient algorithms with a high degree of accuracy, for heterogeneous advection-diffusion problems. The strategy we develop here is to split the time interval into time windows. In each window we will perform a small number of iterations of an optimized Schwarz waveform relaxation algorithm. The subdomain solver is the discontinuous

Galerkin methods in time, and classical finite elements in space. The coupling between the subdomains is done by the extension of a projection procedure written in [6].

After defining our model problem in Section 2, we recall in Section 3 the Schwarz waveform relaxation algorithm, with optimized transmission conditions of order 1 in time, as introduced in [2, 5]. The general discontinuous Galerkin formulation is given in Section 4. In Section 5, we introduce the discrete algorithm in time in the nonconforming case. The projection between arbitrary grids is performed by an efficient algorithm based on the method introduced in [6]. In Section 6, numerical results illustrate the validity of our approach, in particular the superconvergence result proved in [3] for the heat equation and homogeneous Dirichlet boundary conditions is valid.

2 Model Problem

We consider the advection-diffusion problem in $\Omega = (a, b)$:

$$\begin{aligned} &\mathcal{L}u \equiv \frac{\partial u}{\partial t} + \frac{\partial}{\partial x}(a(x)u) - \frac{\partial}{\partial x}(\nu(x)\frac{\partial u}{\partial x}) = f \text{ in } \Omega \times (0,T), \\ &u(\cdot, 0) = u_0 \text{ in } \Omega, \\ &u(a, \cdot) = u(b, \cdot) = 0 \text{ in } (0,T). \end{aligned} \tag{1}$$

The advection coefficient $a(x)$, and the viscosity $\nu(x)$ are in $L^\infty(\Omega)$. ν is bounded from below by a positive constant and we suppose here the advection coefficient to be positive. We are interested in a coupling procedure for a problem with discontinuities in the coefficients, and we suppose a and ν to be continuous in subregions $\Omega_j =]x_{j-1}, x_j[$ of Ω, but possibly discontinuous at interfaces x_j. We shall write

$$a_j^\pm = \lim_{x \to x_j^\pm} a(x), \quad \nu_j^\pm = \lim_{x \to x_j^\pm} \nu(x).$$

Problem (1) is equivalent to finding $\{u_j\}_{j=1,\dots,J}$ solutions of the advection-diffusion equation in each subdomain Ω_j, with the physical transmission conditions in $(0, T)$

$$u_j(x_j^-, \cdot) = u_{j+1}(x_j^+, \cdot), \quad (\nu_j^- \frac{\partial}{\partial x} - a_j^-)u_j(x_j^-, \cdot) = (\nu_j^+ \frac{\partial}{\partial x} - a_j^+)u_{j+1}(x_j^+, \cdot),$$

with $u_j = u_{|\Omega_j}$. In view of applications for long-time computations, we split the time domain into windows and we intend to design an algorithm which requires very few iterations per time window. This will be achieved with an optimized Schwarz waveform relaxation algorithm.

3 Optimized Schwarz Waveform Relaxation with Time Windows

The time interval is divided into time windows, $[0,T] = \cup_{N=0}^{N_w}[T_N, T_{N+1}]$. In each window we perform successively n_N iterations of the Schwarz waveform relaxation algorithm with $n_N \geq 2$ as small a possible, taking as initial value the final value of the last iterate of the algorithm in the previous window. Suppose $\{\tilde{u}_j\}$ is computed that way on $(0, T_N)$. We compute now $\{\tilde{u}_j\}$ on the next window by the algorithm, for $n = 1, ..., n_N$:

$$\mathcal{L}u_j^n = f \text{ in } \Omega_j \times (T_N, T_{N+1}),$$

with the initial value $u_j^n(\cdot, T_N) = \tilde{u}_j(\cdot, T_N^-)$, and the transmission conditions in (T_N, T_{N+1}),

$$(-\nu_{j-1}^+ \frac{\partial}{\partial x} + a_{j-1}^+ Id + S_j^-) u_j^n(x_{j-1}^+, \cdot) = (-\nu_{j-1}^- \frac{\partial}{\partial x} + a_{j-1}^- Id + S_j^-) u_{j-1}^{n-1}(x_{j-1}^-, \cdot),$$

$$(\nu_j^- \frac{\partial}{\partial x} - a_j^- Id + S_j^+) u_j^n(x_j^-, \cdot) = (\nu_j^+ \frac{\partial}{\partial x} - a_j^+ Id + S_j^+) u_{j+1}^{n-1}(x_j^+, \cdot).$$

It was proved in [5, 2] that a suitable choice of the operators $S_j^\pm$, leads to convergence in J iterations. However these "transparent" operators are not easy to handle, and we use instead transmission operators of the form:

$$S_j^- = \frac{p_j^- - a_j^-}{2} Id + \frac{q_j^-}{2}\frac{\partial}{\partial t}, \quad S_j^+ = \frac{a_j^+ + p_j^+}{2} Id + \frac{q_j^+}{2}\frac{\partial}{\partial t}. \tag{2}$$

The initial guesses on the interfaces have to be prescribed, this will be done on the discrete level in Section 6. We then define $\tilde{u}_j$ on (T_N, T_{N+1}) by $u_j^{n_N}$. In order to reduce the number of iterations, we need to make the convergence rate as small as possible. This can be achieved by choosing carefully the parameters $p_j^\pm$ and $q_j^\pm$ such as to minimize the local convergence rate, *i.e.* between two subdomains. Details for the optimization on the theoretical level for continuous coefficients can be found in [1], and for preliminary results in this case see [5]. For positive coefficients $p_j^\pm$ and $q_j^\pm$, the convergence of the algorithm can be proved by the method of energy.

4 Time Discontinuous Galerkin Method

We introduce the discretization of a subproblem in one time window $I = (T_N, T_{N+1})$ and one interval Ω_j. The subproblem at step n of the SWR procedure for an internal subdomain is to find v such that

$$\begin{cases} \mathcal{L}v = f & \text{in } \Omega_j \times I, \\ v(\cdot, T_N) = v_0 & \text{in } \Omega_j, \\ (-\nu^+ \frac{\partial}{\partial x} + \beta^- Id + \gamma^- \frac{\partial}{\partial t})\, v(x_{j-1}^+, \cdot) = g^- & \text{in } I, \\ (\nu^- \frac{\partial}{\partial x} + \beta^+ Id + \gamma^+ \frac{\partial}{\partial t})\, v(x_j^-, \cdot) = g_j^+ & \text{in } I. \end{cases} \tag{3}$$

Subproblems at either end of the interval have one boundary condition replaced by a Dirichlet boundary condition. Let $V_j = H^1(\Omega_j)$. This problem has the weak formulation: find v in $\mathcal{C}^0(0,T;L^2(\Omega_j)) \cap L^2(0,T;H^1(\Omega_j))$ such that $v(\cdot,T_N) = v_0$ and

$$(\!(\frac{dv}{dt}(t),\varphi)\!) + b(v(t),\varphi) = (f(t),\varphi) + g^-(t)\varphi(x_{j-1}) + g^+(t)\varphi(x_j), \quad \forall \varphi \in V_j,$$

with $(\cdot,\cdot)$ the scalar product in $L^2(\Omega_j)$, and for φ,ψ in V_j:

$$\begin{cases} (\!(\varphi,\psi)\!) = (\varphi,\psi) + \gamma^-\varphi(x_{j-1})\psi(x_{j-1}) + \gamma^+\varphi(x_j)\psi(x_j), \\ b(\varphi,\psi) = (\nu(x)\dfrac{\partial\varphi}{\partial x},\dfrac{\partial\psi}{\partial x}) + (a(x)\varphi,\dfrac{\partial\psi}{\partial x}) \\ \qquad\qquad + \beta^-\varphi(x_{j-1})\psi(x_{j-1}) + \beta^+\varphi(x_j)\psi(x_j). \end{cases}$$

For positive $\beta^\pm$ and $\gamma^\pm$, this problem is well-posed in suitable Sobolev spaces, see [1]. For the discretization of (3) in time, we use the discontinuous Galerkin method [7] which is a Galerkin method with discontinuous piecewise polynomials of degree $q \geq 0$ defined as follows. Let $\mathcal{T}$ be a decomposition of I into $I = \cup_{k=1}^K I_k$ with $I_k = [t_{k-1},t_k]$, and $\Delta t_k = t_k - t_{k-1}$. For any space V, we define

$$\mathbf{P}_q(V) = \{\varphi : I \to V,\ \varphi(t) = \sum_{i=0}^q \varphi_i t^i,\ \varphi_i \in V\}$$
$$\mathcal{P}_q(V,\mathcal{T}) = \{\varphi : I \to V,\ \varphi_{|I_k} \in \mathbf{P}_q(V),\ 1 \leq k \leq K\}.$$

As the functions in $\mathcal{P}_q(V,\mathcal{T})$ may be discontinuous at the mesh points t_k, we define $\varphi^{k,\pm} = \varphi(t_k \pm 0)$. The discontinuous Galerkin Method, as formulated in [7], defines recursively on the intervals I_k, an approximate solution U of (4) in $\mathcal{P}_q(V_j,\mathcal{T})$, by

$$\begin{cases} U^{0,-} = v_0, \\ \forall \varphi \in \mathbf{P}_q(V_j) : \quad \displaystyle\int_{I_k} [(\!(\frac{dU}{dt},\varphi)\!) + b(U,\varphi)]\,dt + ((U^{k-1,+},\varphi^{k-1,+})) = \\ \displaystyle\int_{I_k} [(f(t),\varphi(t)) + g^-(t)\varphi(x_{j-1}) + g^+(t)\varphi(x_j)]dt + ((U^{k-1,-},\varphi^{k-1,+})). \end{cases} \tag{4}$$

Theorem 1. *For $f \in L^\infty(0,T;L^2(\Omega_j))$, $g^\pm \in L^\infty(0,T)$, and $\beta^\pm,\gamma^\pm > 0$, equation (4) has a unique solution $U \in \mathcal{P}_q(V_j,\mathcal{T})$. Moreover, for sufficiently smooth v, we have the error estimate:*

$$\|v-U\|_{L^\infty(I;L^2(\Omega_j))} \leq C^i C_q^s (\max_{1\leq k\leq K} L_k) \|\Delta t^{q+1} v^{(q+1)}\|_{L^\infty(I;L^2(\Omega_j))}, \tag{5}$$

where Δt is the local time step defined in I_k by $\Delta t(s) = \Delta t_k$, C_q^s is a stability constant related to the dG-discretization, independent of T, u, Δt, and U. C^i is an interpolation constant depending only on q, and $L_k = 1 + log(t_k/\Delta t_k)$.

Proof. Let $H = L^2(\Omega_j) \times \mathbb{R} \times \mathbb{R}$ with inner product $(\cdot,\cdot)_H$ defined for $U_1 = (\varphi_1, \alpha_1^-, \alpha_1^+)$ and $U_2 = (\varphi_2, \alpha_2^-, \alpha_2^+)$ in H by $(U_1, U_2)_H = (\varphi_1, \varphi_2) + \gamma_j^- \alpha_1^- \alpha_2^- + \gamma_j^+ \alpha_1^+ \alpha_2^+$. Let $D(A) = \{U = (\varphi, \alpha^-, \alpha^+),\ \varphi \in H^2(\Omega_j),\ \varphi(x_{j-1}) = \alpha^-,\ \varphi(x_j) = \alpha^+\}$. Let $A : D(A) \subset H \to H$ defined by

$$A = \begin{pmatrix} -\dfrac{\partial}{\partial x}(\nu \dfrac{\partial}{\partial x}) + \dfrac{\partial}{\partial x}(a\ Id) & 0 & 0 \\ \dfrac{\nu_{j-1}^+}{\gamma_j^-} \dfrac{\partial}{\partial x} & \dfrac{\beta_j^-}{\gamma_j^-} & 0 \\ \dfrac{\nu_j^-}{\gamma_j^+} \dfrac{\partial}{\partial x} & 0 & \dfrac{\beta_j^+}{\gamma_j^+} \end{pmatrix}$$

Then, the proof of Theorem 1 is based on the theoretical result in [4], since A is the infinitesimal generator of an analytic, uniformly bounded semi-group.

Remark 1. An analogous result holds in higher dimension and for general boundaries, as soon as they do not intersect.

5 The Discontinuous Galerkin Schwarz Waveform Relaxation

Let us consider the case where the time steps are different in the subdomains: in each Ω_j, let $\mathcal{T}_j$ be a partition of the time interval into $I = \cup_{k=1}^K I_k^j$ with $I_k^j = [t_{k-1}^j, t_k^j]$. Then, we need a projection procedure to transfer the boundary values from one domain to his two neighbors. We now define the precise procedure in domain Ω_j. Let $(g_j^{-,n-1}, g_j^{+,n-1})$ be given in $\mathcal{P}_q(\mathbb{R}, \mathcal{T}_j)$. Then, one iteration of the SWR method consists in the following steps:

$$\begin{array}{ccc} g_j^{-,n-1} \in \mathcal{P}_q(\mathbb{R}, \mathcal{T}_j) & & g_j^{+,n-1} \in \mathcal{P}_q(\mathbb{R}, \mathcal{T}_j) \\ \searrow & & \swarrow \\ & U_j^n \in \mathcal{P}_q(V_j, \mathcal{T}_j) & \\ \swarrow & & \searrow \\ \tilde{g}_j^{-,n} = (\nu_{j-1}^+ \partial_x - a_{j-1}^+ + S_{j-1}^+) U_j^n(x_{j-1}^+, \cdot) & & \tilde{g}_j^{+,n} = (-\nu_j^- \partial_x + a_j^- - S_{j+1}^-) U_j^n(x_j^-, \cdot) \\ \tilde{g}_j^{-,n} \in \mathcal{P}_q(\mathbb{R}, \mathcal{T}_j) & & \tilde{g}_j^{+,n} \in \mathcal{P}_q(\mathbb{R}, \mathcal{T}_j) \\ \downarrow & & \downarrow \\ g_{j-1}^{+,n} = P_{j,j-1} \tilde{g}_j^{-,n} \in \mathcal{P}_q(\mathbb{R}, \mathcal{T}_{j-1}) & & g_{j+1}^{-,n} = P_{j,j+1} \tilde{g}_j^{+,n} \in \mathcal{P}_q(\mathbb{R}, \mathcal{T}_{j+1}) \end{array}$$

U_j^n is the solution of (4) in Ω_j with coefficients $\beta_j^\pm$ and $\gamma_j^\pm$ given by (2), and data $(g_j^{-,n-1}, g_j^{+,n-1})$. $P_{i,j}$ is the orthogonal L^2 projection on $\mathcal{P}_q(\mathbb{R}, \mathcal{T}_j)$, restricted to $\mathcal{P}_q(\mathbb{R}, \mathcal{T}_i)$.

Note that the computations in different subdomains on the same time window (T_N, T_{N+1}) can be done in parallel. One could even think of using multigrid in time or asynchronous algorithm.

6 Numerical Results

In order to see the effect of the coupling of discontinuous Galerkin with the domain decomposition algorithm, we perform numerical simulations with two subdomains only. We choose ν and a to be constant in each subdomain. For the space discretization, we replace V_j by a finite-dimensional subspace V_j^h (standard $\mathbb{P}_1$ finite element space) of V_j in (4).
Our computations are performed with $q = 1$ in the discontinuous Galerkin method. In that case a superconvergence result was proved in [3] for the heat equation and homogeneous Dirichlet boundary conditions: under suitable assumptions on the space and time steps, the accuracy is of order 3 in time at the discrete time levels t_k: let $\| \cdot \|_{k,j} = \| \cdot \|_{L^\infty(I_k;L^2(\Omega_j))}$,

$$\|v(t_k) - U^{k,-}\|_{L^2(\Omega_j)} \leq C_k \max_{1\leq k\leq K} \left\{ \min_{0\leq \ell\leq 3} \Delta t_k^\ell \|\partial_t^{(\ell)} v\|_{k,j} + \min_{1\leq \ell\leq 2} h^\ell \|D^\ell v\|_{k,j} \right\}, \quad (6)$$

where h is the mesh size, $C_k = C(L_k)^{\frac{1}{2}}$ with C a constant independent of T, v, Δt, and U. Our numerical results will illustrate both estimates (5) and (6).

In the sequel, we denote by "1-window converged solution", the iterate of the optimized Schwarz waveform relaxation algorithm in one time window (the whole time interval), for which the residual on the boundary (*i*.e. $\|g_j^{\pm,n} - g_j^{\pm,n-1}\|$) is smaller than 10^{-8}.

6.1 An Example of Multidomain Solution with Time Windows

In this part, we consider Problem (1) on $\Omega =]0, 6[$ with $f \equiv 0$, and the final time is $T = 2$. The initial value is $u_0 = e^{-3(2.5-x).^2}$. Ω is split into two subdomains $\Omega_1 = (0, 3)$ and $\Omega_2 = (3, 6)$. In each subdomain the advection and viscosity coefficients are constant, equal to $a_1 = 0.1$, $\nu_1 = 0.2$, $a_2 = \nu_2 = 1$. The mesh size is $h = 0.06$ for each subdomain. The number of time grid points in each window is $n_1 = 61$ for Ω_1, and $n_2 = 25$ for Ω_2. We denote by "6-windows solution", the approximate solution computed using 6 time windows, with $n_N = 2$ iterations of the optimized Schwarz waveform relaxation algorithm in each time window. The initial guess $g_j^{\pm,0}$ on the interface in time window (T_N, T_{N+1}) is given at time T_N by the exact discrete value in the previous window, and is taken to be constant on the time interval. In Figure 1 we observe that at the final time $T = 2$, the 6-windows solution and the 1-window converged solution cannot be distinguished. Since the cost of the computation is much less with time windows, this validates the approach.

6.2 Error Estimates

Constant Coefficients

We first analyze the error in (4) for constant coefficients $a \equiv 1, \nu \equiv 1$. The exact solution is $u(x,t) = \cos(x)\cos(t)$, in $[-\pi/2, \pi/2] \times [0, 2\pi]$. The interface is at $x = 1$. The mesh size is $h = \pi \cdot 10^{-4}$, for each subdomain. The time steps are initially $\Delta t_1 = 2\pi/4$ in Ω_1 and $\Delta t_2 = 2\pi/6$ in Ω_2, and thereafter divided by 2 several times. Let $\Delta t = \max(\Delta t_1, \Delta t_2)$. Figure 2 shows the norms of the error involved in the estimates (5) and (6). The numerical results fit the theoretical estimates.

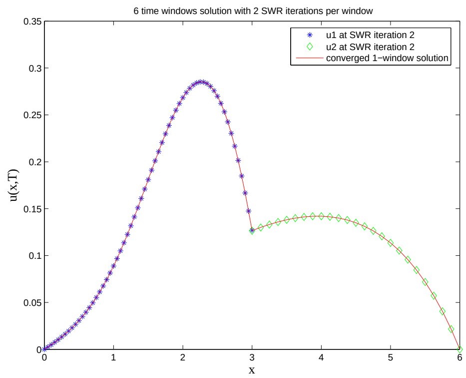

Fig. 1. 1-window converged solution (solid line) and 6-windows solution (star line for Ω_1 and diamond line for Ω_2), at time t=T=2.

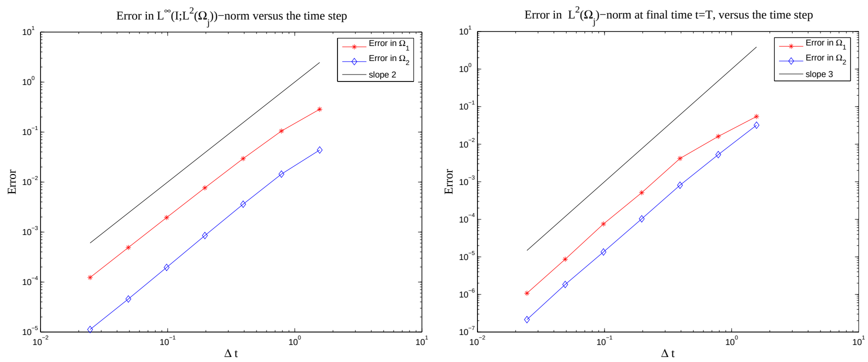

Fig. 2. Error in $L^\infty(I; L^2(\Omega_j))$-norm (on the left) and in $L^2(\Omega_j)$-norm at the final time $t = T$ (on the right), versus the time step Δt, for Ω_1 (star line) and for Ω_2 (diamond line), in the case of constant coefficients.

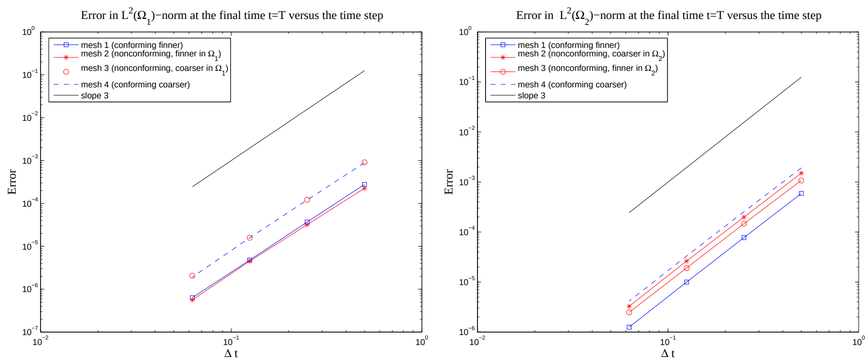

Fig. 3. $L^2(\Omega_j)$ error at the final time $t = T$ versus the time step Δt, for Ω_1 (on the left) and Ω_2 (on the right), for the meshes *Mesh 1,2,3,4* , in the case of discontinuous coefficients.

Discontinuous Coefficients

We consider the configuration in Section 6.1 with one time window. We observe the error between the 1-window converged solution and a reference solution (the 1-window converged solution on a very fine space-time grid), versus the time step. The mesh size is $h = 3 \cdot 10^{-4}$ for each subdomain. We consider four initial meshes in time

- *Mesh 1*: a uniform conforming finner grid with $\Delta t_1 = \Delta t_2 = T/6$,
- *Mesh 2*: a nonconforming grid with $\Delta t_1 = T/6$ and $\Delta t_2 = T/4$,
- *Mesh 3*: a nonconforming grid with $\Delta t_1 = T/4$ and $\Delta t_2 = T/6$,
- *Mesh 4*: a uniform conforming coarser grid with $\Delta t_1 = \Delta t_2 = T/4$.

Thereafter $\Delta t_j,\ j = 1, 2$, is divided by two at each computation. Figure 3 shows the error versus the time step $\Delta t = \max(\Delta t_1, \Delta t_2)$, for these four meshes, in Ω_1 (on the left) and in Ω_2 (on the right). The results show that the $L^2(\Omega_j)$ error at discrete time points tends to zero at the same rate as Δt^3, and this fits with the error estimate (6). On the other hand, we observe that the two curves corresponding to the nonconforming meshes are between the curves of the conforming meshes. We obtain similar results for the $L^\infty(I; L^2(\Omega_j))$ error, which fits with (5).

7 Conclusions

We have proposed a new numerical method for the advection-diffusion equation with discontinuous coefficients. It relies on the splitting of the time interval into time windows. In each window a few iterations of a Schwarz waveform relaxation algorithm are performed by a discontinuous Galerkin method, with projection of the time-grids on the interfaces of the spacial subregions. We have shown numerically that our method preserves the order of the discontinuous Galerkin method.

References

[1] D. Bennequin, M.J. Gander, and L. Halpern. Optimized Schwarz Waveform Relaxation for convection reaction diffusion problems. Technical Report 24, LAGA, Université Paris 13, 2004.

[2] E. Blayo, L. Halpern, and C. Japhet. Optimized Schwarz Waveform Relaxation algorithms with nonconforming time discretization for coupling convection-diffusion problems with discontinuous coefficients. In *Domain Decomposition Methods in Science and Engineering XVI*, pages 267–274. Springer, Berlin, 2007.

[3] K. Eriksson and C. Johnson. Adaptive finite element methods for parabolic problems II: optimal error estimates in $L_\infty L_2$ and $L_\infty L_\infty$. *SIAM J. Numer. Anal.*, 32(3):706–740, 1995.

[4] K. Eriksson, C. Johnson, and S. Larsson. Adaptive finite element methods for parabolic problems VI: analytic semigroups. *SIAM J. Numer. Anal.*, 35(4):1315–1325, 1998.

[5] M. Gander, L. Halpern, and M. Kern. Schwarz Waveform Relaxation method for advection–diffusion–reaction problems with discontinuous coefficients and non-matching grids. In *Domain Decomposition Methods in Science and Engineering XVI*, pages 283–290, Berlin, 2007. Springer.

[6] M.J. Gander, L. Halpern, and F. Nataf. Optimized Schwarz Waveform Relaxation method for the one dimensional wave equation. *SIAM J. Numer. Anal.*, 41(5), 2003.

[7] C. Johnson. *Numerical Solution of Partial Differential Equations by the Finite Element Method*. Cambridge University Press, New York, 1987.

[8] V. Martin. An optimized Schwarz Waveform Relaxation method for unsteady convection diffusion equation. *Appl. Numer. Math.*, 52(4):401–428, 2005.