Three-level BDDC

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Summary. BDDC (Balancing Domain Decomposition by Constraints) methods, so far developed for two levels [3, 7, 8], are similar to the balancing Neumann-Neumann algorithms. However, the BDDC coarse problem is given in terms of a set of primal constraints and the matrix of the coarse problem is generated and factored by direct solvers at the beginning of the computation. The coarse component of the preconditioner can ultimately become a bottleneck if the number of subdomains is very large. In this paper, two three-level BDDC methods are introduced for solving the coarse problem approximately in two and three dimensions, while still maintaining a good convergence rate. Estimates of the condition numbers are provided for the two three-level BDDC methods and numerical experiments are also discussed.

1 The two-level BDDC method

We consider a second order scalar elliptic problem in a two or three dimensional region Ω: find $u \in H_0^1(\Omega)$, such that

$$\int_\Omega \rho \nabla u \cdot \nabla v = \int_\Omega f v \quad \forall v \in H_0^1(\Omega), \tag{1}$$

where $\rho(x) > 0$ for all $x \in \Omega$. We introduce a mesh, subdomains Ω_i, and an interface Γ on the domain Ω with notation as in [10, Section 4.2].

Let $\mathbf{W}^{(i)}$ be the standard conforming first order finite elements on Ω_i. We assume that these functions vanish on $\partial\Omega$. Each $\mathbf{W}^{(i)}$ can be decomposed into a subdomain interior part $\mathbf{W}_I^{(i)}$ and a subdomain interface part $\mathbf{W}_\Gamma^{(i)}$. The subdomain interface part $\mathbf{W}_\Gamma^{(i)}$ can be further decomposed into a primal subspace $\mathbf{W}_\Pi^{(i)}$ and a dual subspace $\mathbf{W}_\Delta^{(i)}$, i.e., $\mathbf{W}^{(i)} = \mathbf{W}_I^{(i)} \bigoplus \mathbf{W}_\Gamma^{(i)} = \mathbf{W}_I^{(i)} \bigoplus \mathbf{W}_\Pi^{(i)} \bigoplus \mathbf{W}_\Delta^{(i)}$.

We denote the associated product spaces by $\mathbf{W} := \prod_{i=1}^{N} \mathbf{W}^{(i)}$, $\mathbf{W}_\Gamma := \prod_{i=1}^{N} \mathbf{W}_\Gamma^{(i)}$, $\mathbf{W}_\Delta := \prod_{i=1}^{N} \mathbf{W}_\Delta^{(i)}$, $\mathbf{W}_\Pi := \prod_{i=1}^{N} \mathbf{W}_\Pi^{(i)}$, and $\mathbf{W}_I := \prod_{i=1}^{N} \mathbf{W}_I^{(i)}$. Correspondingly, we have $\mathbf{W} = \mathbf{W}_I \bigoplus \mathbf{W}_\Gamma$ and $\mathbf{W}_\Gamma = \mathbf{W}_\Pi \bigoplus \mathbf{W}_\Delta$.

We will consider elements of the product space $\mathbf{W}$ which are discontinuous across the interface. However, the finite element approximation of the elliptic problem is continuous across Γ and we denote the corresponding subspace of $\mathbf{W}$ by $\widehat{\mathbf{W}}$.

We further introduce an interface subspace $\widetilde{\mathbf{W}}_\Gamma \subset \mathbf{W}_\Gamma$, for which certain primal constraints are enforced. Here, the continuous primal subspace, denoted by $\widehat{\mathbf{W}}_\Pi$, is only spanned by the continuous finite element basis functions of the vertex nodes in two dimensions and by the continuous edge average variables over each subdomain edge in three dimensions. For three dimensions, we change the variables to make the edge average degrees of freedom explicit, see [5, Sec 6.2] and [6, Sec 2.3]. From now on, we assume that all the matrices are written in terms of the new variables in three dimensions. The space $\widetilde{\mathbf{W}}_\Gamma$ can be decomposed into $\widetilde{\mathbf{W}}_\Gamma = \widehat{\mathbf{W}}_\Pi \bigoplus \mathbf{W}_\Delta$.

We define an operator $\widetilde{S}_\Gamma$ by: given $\mathbf{u}_\Gamma = \mathbf{u}_\Pi \oplus \mathbf{u}_\Delta \in \widehat{\mathbf{W}}_\Pi \bigoplus \mathbf{W}_\Delta = \widetilde{\mathbf{W}}_\Gamma$, we find $\widetilde{S}_\Gamma \mathbf{u}_\Gamma$ by eliminating the interior variables of the partially assembled system with continuous primal components.

The operator $R_{\Gamma\Delta} : \widetilde{\mathbf{W}}_\Gamma \rightarrow \mathbf{W}_\Delta$, restricts the functions in the space $\widetilde{\mathbf{W}}_\Gamma$ to $\mathbf{W}_\Delta$, and is a block diagonal matrix $diag(R_{\Gamma\Delta}^{(1)}, \cdots, R_{\Gamma\Delta}^{(N)})$, where each $R_{\Gamma\Delta}^{(i)}$ represents the restriction from $\mathbf{W}_\Gamma^{(i)}$ to $\mathbf{W}_\Delta^{(i)}$. Furthermore, $R_\Delta^{(i)} : \mathbf{W}_\Delta \rightarrow \mathbf{W}_\Delta^{(i)}$, is the restriction matrix which extracts the subdomain part, in the space $\mathbf{W}_\Delta^{(i)}$, of the functions in the space $\mathbf{W}_\Delta$, and $R_{\Gamma\Pi}$ restricts the functions in the space $\widetilde{\mathbf{W}}_\Gamma$ to $\widehat{\mathbf{W}}_\Pi$. $R_\Pi^{(i)}$ is the restriction operator from the space $\widehat{\mathbf{W}}_\Pi$ to $\mathbf{W}_\Pi^{(i)}$.

$R_\Gamma = (R_\Gamma^{(1)}, \cdots, R_\Gamma^{(N)})^T$ and $R_{D,\Gamma} = (R_{D,\Gamma}^{(1)}, \cdots, R_{D,\Gamma}^{(N)})^T$ are the restriction and scaled restriction operators from the space $\widehat{\mathbf{W}}_\Gamma$ onto $\widetilde{\mathbf{W}}_\Gamma$, respectively. Here $R_\Gamma^{(i)}$ maps a vector in $\widehat{\mathbf{W}}_\Gamma$ to a vector in $\mathbf{W}_\Gamma^{(i)}$. Each column of $R_\Gamma^{(i)}$ with a nonzero entry corresponds to an interface node, $x \in \partial\Omega_{i,h} \cap \Gamma_h$, shared by the subdomain Ω_i and its neighboring subdomains. Multiplying each such column of $R_\Gamma^{(i)}$ with $\delta_i^\dagger(x)$ gives us $R_{D,\Gamma}^{(i)}$, where $\delta_i^\dagger(x)$ is defined in [10, Formula (6.2)].

The reduced interface problem can be written as: find $\mathbf{u}_\Gamma \in \widehat{\mathbf{W}}_\Gamma$ such that $R_\Gamma^T \widetilde{S}_\Gamma R_\Gamma \mathbf{u}_\Gamma = \mathbf{g}_\Gamma$, where $\mathbf{g}_\Gamma$ is the load vector reduced to Γ.

The two-level BDDC preconditioned equation is of the form

$$M^{-1} R_\Gamma^T \widetilde{S}_\Gamma R_\Gamma \mathbf{u}_\Gamma = M^{-1} \mathbf{g}_\Gamma,$$

where the preconditioner $M^{-1} = R_{D,\Gamma}^T \widetilde{S}_\Gamma^{-1} R_{D,\Gamma}$ has the following form (see [6, Formula (33)]) with the columns of Φ, being minimal energy extensions of the primal variables:

$$R_\Gamma^T D_\Gamma \left\{ \sum_{i=1}^{N} R_{\Gamma\Delta}^{(i)^T} \begin{pmatrix} \mathbf{0} & R_\Delta^{(i)^T} \end{pmatrix} \begin{pmatrix} A_{II}^{(i)} & A_{\Delta I}^{(i)} \\ A_{\Delta I}^{(i)} & A_{\Delta\Delta}^{(i)} \end{pmatrix}^{-1} \begin{pmatrix} \mathbf{0} \\ R_\Delta^{(i)} \end{pmatrix} R_{\Gamma\Delta}^{(i)} + \Phi S_\Pi^{-1} \Phi^T \right\} D_\Gamma R_\Gamma.$$

Denote by E_D and P_D, the average and jump operators (see [10, Formula (6.4) and (6.38)]), on the space $\widetilde{\mathbf{W}}_\Gamma$, respectively. Central to obtaining the condition number estimate for the preconditioned two-level BDDC operator is a bound for the E_D operator (see [8, Theorem 25]). Since $E_D + P_D = I$ (see [10, Lemma 6.10]), we only need to find a bound for the P_D operator.

A bound for the P_D operator in two dimensions is given in [9], provided that the coefficient $\rho(x)$ of (1) varies moderately in each subdomain. In our theory, we also assume that each subdomain is a union of shape-regular coarse triangles and that the number of such triangles forming an individual subdomain is uniformly bounded. Moreover, we assume that the triangulation of each subdomain is quasi uniform. For the three dimensional case, we need one more requirement for $\rho(x)$ since we only use the edge average constraints, namely that for all pairs of subdomain Ω_i and Ω_j, which have a vertex but not an edge in common, there exists an acceptable edge path (see [10, Definition 6.26]) between the two subdomains. With this assumption, we have a good estimate for the P_D operator (see [10, Lemma 6.36]): under our assumptions, we have in two and three dimensions:

$$\mathbf{u}_\Gamma^T M \mathbf{u}_\Gamma \leq \mathbf{u}_\Gamma^T R_\Gamma^T \widetilde{S}_\Gamma R_\Gamma \mathbf{u}_\Gamma \leq C \left(1 + \log(H/h)\right)^2 \mathbf{u}_\Gamma^T M \mathbf{u}_\Gamma, \ \forall \mathbf{u}_\Gamma \in \widehat{\mathbf{W}}_\Gamma.$$

2 A three-level BDDC method

In the three-level case, we will not factor the coarse problem matrix S_Π by a direct solver. Instead, we will solve the coarse problem approximately by using an idea similar to the two-level preconditioner.

We decompose Ω into N subregions Ω^j with diameters $\hat{H}^j$, $j = 1, \cdots, N$. Each subregion Ω^j has N_j subdomains Ω_i^j with diameter H_i^j. Let $\hat{H} = \max_j \hat{H}^j$ and $H = \max_{i,j} H_i^j$, for $j = 1, \cdots, N$, and $i = 1, \cdots, N_j$. We introduce the subregional Schur complements:

$$S_\Pi^{(j)} = \sum_{i=1}^{N_j} R_\Pi^{(i)^T} \left\{ A_{\Pi\Pi}^{(i)} - \begin{pmatrix} A_{\Pi I}^{(i)} & A_{\Pi\Delta}^{(i)} \end{pmatrix} \begin{pmatrix} A_{II}^{(i)} & A_{I\Delta}^{(i)} \\ A_{\Delta I}^{(i)} & A_{\Delta\Delta}^{(i)} \end{pmatrix}^{-1} \begin{pmatrix} A_{\Pi I}^{(i)^T} \\ A_{\Pi\Delta}^{(i)^T} \end{pmatrix} \right\} R_\Pi^{(i)},$$

and note that the coarse problem matrix S_Π can be assembled from the $S_\Pi^{(j)}$.

Let $\widehat{\Gamma}$ be the interface between the subregions; $\widehat{\Gamma} \subset \Gamma$. We denote the set of interior primal variables in each subregion by $\widehat{I}_H$, and the set of interface primal variables on the boundary of the subregions by $\widehat{\Gamma}_H$.

We denote the vector space corresponding to the primal variables of the subregion Ω^i by $\mathbf{W}_c^{(i)}$. We define the subregion spaces $\widehat{\mathbf{W}}_{c,\widehat{\Gamma}}$, $\widetilde{\mathbf{W}}_{c,\widehat{\Gamma}}$, $\widehat{R}_{\widehat{\Gamma}}^{(i)}$, $\widehat{R}_{\widehat{D},\widehat{\Gamma}}^{(i)}$, $\widehat{R}_{\widehat{\Gamma}}$, and $\widehat{R}_{\widehat{D},\widehat{\Gamma}}$, as for the subdomains but on the subregion level.

We introduce an operator $\widetilde{T}$:

$$\widehat{R}_{\widehat{\Gamma}}^T \widetilde{T} \widehat{R}_{\widehat{\Gamma}} = \sum_{i=1}^{N} \widehat{R}_{\widehat{\Gamma}}^{(i)^T} (S_{\Pi\widehat{\Gamma}\widehat{\Gamma}}^{(i)} - S_{\Pi\widehat{\Gamma}I}^{(i)} S_{\Pi\widehat{I}I}^{(i)^{-1}} S_{\Pi\widehat{I}\widehat{\Gamma}}^{(i)^T}) \widehat{R}_{\widehat{\Gamma}}^{(i)}. \tag{2}$$

and define our three-level preconditioner $\widetilde{M}^{-1}$ by

$$R_\Gamma^T D_\Gamma \left\{ \sum_{i=1}^{N} R_{\Gamma\Delta}^{(i)^T} \begin{pmatrix} \mathbf{0} & R_\Delta^{(i)^T} \end{pmatrix} \begin{pmatrix} A_{II}^{(i)} & A_{\Delta I}^{(i)} \\ A_{\Delta I}^{(i)} & A_{\Delta\Delta}^{(i)} \end{pmatrix}^{-1} \begin{pmatrix} \mathbf{0} \\ R_\Delta^{(i)} \end{pmatrix} R_{\Gamma\Delta}^{(i)} + \Phi \widetilde{S}_\Pi^{-1} \Phi^T \right\} D_\Gamma R_\Gamma.$$

Here $\widetilde{S}_\Pi^{-1}$ is an approximation of S_Π^{-1} and is defined as follows: given any right hand side $\mathbf{\Psi}$, let $\mathbf{y} = S_\Pi^{-1}\mathbf{\Psi}$ and $\widetilde{\mathbf{y}} = \widetilde{S}_\Pi^{-1}\mathbf{\Psi}$. We first reduce the original coarse problem S_Π to the subregion interface problem. We do not solve the interface problem exactly but replace $\mathbf{y}_{\widehat{\Gamma}}$, the interface part of $\mathbf{y}$, by

$$\widetilde{\mathbf{y}}_{\widehat{\Gamma}} = \widehat{R}_{\widehat{D},\widehat{\Gamma}}^T \widetilde{T}^{-1} \widehat{R}_{\widehat{D},\widehat{\Gamma}} \mathbf{h}_{\widehat{\Gamma}},$$

where $\mathbf{h}_{\widehat{\Gamma}}$ is the load vector reduced to $\widehat{\Gamma}$.

3 Condition number estimate for the new preconditioner

We first collect a number of results which are needed in our theory. We discuss, in detail, only the two dimensional case.

Lemma 1. *(Two dimensions) Let V_i^H be the standard continuous piecewise linear finite element function space for a subregion Ω^i with a quasi-uniform coarse mesh with mesh size H. And let $V_{i,j}^h$, $j = 1, \cdots, N_i$ be the space for a subdomain Ω_j^i with a quasi-uniform fine mesh with mesh size h. Moreover, each subdomain is a union of coarse triangles with vertices on the boundary of the subdomain. Given $u \in V_i^H$, let $\hat{u} \in V_i^H$ interpolate u at each coarse node and be the discrete $V_{i,j}^h$-harmonic extension in each subdomain Ω_j^i constrained only at the vertices of Ω_j^i, $j = 1, \cdots, N_i$. Then, there exist two positive constants C_1 and C_2, which are independent of $\hat{H}$, H, and h, such that*

$$C_1(1+\log\frac{H}{h}) \left(\sum_{j=1}^{N_i} |\hat{u}|_{H^1(\Omega_j^i)}^2 \right) \leq |u|_{H^1(\Omega^i)}^2 \leq C_2(1+\log\frac{H}{h}) \left(\sum_{j=1}^{N_i} |\hat{u}|_{H^1(\Omega_j^i)}^2 \right).$$

We use [2, Lemma 4.2] to prove Lemma 1. Since we assume that the fine triangulation of each subdomain is quasi uniform, we can then obtain uniform constants C_1 and C_2 in Lemma 1 which work for all the subregions. In addition, a similar result for three dimensions can be obtained with [1, Lemma 4.2].

We define the subregion interface averages operator $\widehat{E}_{\widehat{D}} : \widetilde{\mathbf{W}}_{c,\widehat{\Gamma}} \rightarrow \widehat{\mathbf{W}}_{c,\widehat{\Gamma}}$, by $\widehat{E}_{\widehat{D}} = \widehat{R}_{\widehat{\Gamma}}\widehat{R}^T_{\widehat{D},\widehat{\Gamma}}$, which computes averages across the subregion interface $\widehat{\Gamma}$ and then distributes the averages to the boundary points of the subregions.

The interface average operator $\widehat{E}_{\widehat{D}}$ has the following properties:

Lemma 2.

$$\widehat{E}_{\widehat{D}}\mathbf{w}_{\widehat{\Gamma}} = \widehat{R}^T_{\widehat{\Gamma}}\widehat{R}_{\widehat{D},\widehat{\Gamma}}\mathbf{w}_{\widehat{\Gamma}} = \mathbf{w}_{\widehat{\Gamma}}, \text{ for any } \mathbf{w}_{\widehat{\Gamma}} \in \widehat{\mathbf{W}}_{c,\widehat{\Gamma}}.$$

Lemma 3.

$$|\widehat{E}_{\widehat{D}}\mathbf{w}_{\widehat{\Gamma}}|^2_{\widetilde{T}} \leq C\left(1+\log\frac{\hat{H}}{H}\right)^2 |\mathbf{w}_{\widehat{\Gamma}}|^2_{\widetilde{T}},$$

for any $\mathbf{w}_{\widehat{\Gamma}} \in \widetilde{\mathbf{W}}_{c,\widehat{\Gamma}}$, *where* C *is a positive constant independent of* $\hat{H}$, H, *and* h. *Here* $\widetilde{T}$ *is defined in (2).*

See [11] for a proof in two dimensions and [12] for a proof in three dimensions. As we mentioned before, we use constraints on the averages over edges in three dimensions. These constraints lead to a considerably more complicated coarse problem which needs new technical tools in the proof of Lemma 3. This is the main difference in the analysis between two and three dimensions.

Lemma 4. *Given any* $\mathbf{u}_{\mathbf{\Gamma}} \in \widehat{\mathbf{W}}_{\Gamma}$, *let* $\mathbf{\Psi} = \Phi^T D_\Gamma R_\Gamma \mathbf{u}_\Gamma$. *We have,*

$$\mathbf{\Psi}^T S_\Pi^{-1}\mathbf{\Psi} \leq \mathbf{\Psi}^T \widetilde{S}_\Pi^{-1}\mathbf{\Psi}^T \leq C\left(1+\log\frac{\hat{H}}{H}\right)^2 \mathbf{\Psi}^T S_\Pi^{-1}\mathbf{\Psi}.$$

Lemma 5. *Given any* $\mathbf{u}_{\mathbf{\Gamma}} \in \widehat{\mathbf{W}}_{\Gamma}$,

$$\mathbf{u}_\Gamma^T M^{-1}\mathbf{u}_\Gamma \leq \mathbf{u}_\Gamma^T \widetilde{M}^{-1}\mathbf{u}_\Gamma \leq C\left(1+\log\frac{\hat{H}}{H}\right)^2 \mathbf{u}_\Gamma^T M^{-1}\mathbf{u}_\Gamma.$$

We finally have

Theorem 1. *The condition number for the system with the three-level preconditioner* $\widetilde{M}^{-1}$ *is bounded by* $C(1+\log\frac{\hat{H}}{H})^2(1+\log\frac{H}{h})^2$.

4 Using Chebyshev iterations

Another approach to the three-level BDDC methods is to use a preconditioned Chebyshev method with a fixed number of iterations to solve the reduced coarse level subregion interface problem. The preconditioner is $\widehat{R}^T_{\widehat{D},\widehat{\Gamma}}\widetilde{T}^{-1}\widehat{R}_{\widehat{D},\widehat{\Gamma}}$. Denoting the corresponding new coarse problem matrix by $\widehat{S}_H$, the new preconditioner $\widehat{M}^{-1}$ is defined by:

$$R_\Gamma^T D_\Gamma \left\{ \sum_{i=1}^{N} R_{\Gamma\Delta}^T \begin{pmatrix} \mathbf{0} & R_\Delta^{(i)^T} \end{pmatrix} \begin{pmatrix} A_{II}^{(i)} & A_{\Delta I}^{(i)} \\ A_{\Delta I}^{(i)} & A_{\Delta\Delta}^{(i)} \end{pmatrix}^{-1} \begin{pmatrix} \mathbf{0} \\ R_\Delta^{(i)} \end{pmatrix} R_{\Gamma\Delta} + \Phi \widehat{S}_\Pi^{-1} \Phi^T \right\} D_\Gamma R_\Gamma.$$

Denoting by λ_j the eigenvalues of $\left(\widehat{R}_{\widehat{D},\widehat{\Gamma}}^T \widetilde{T}^{-1} \widehat{R}_{\widehat{D},\widehat{\Gamma}}\right)\left(\widehat{R}_{\widehat{\Gamma}}^T \widetilde{T} \widehat{R}_{\widehat{\Gamma}}\right)$, we need two input parameters l and u for the Chebyshev iterations, where l and u are estimates for the minimum and maximum values of λ_j, respectively, see [4]. From our analysis above, we know that $\min_j \lambda_j = 1$ and $\max_j \lambda_j \leq C(1 + \log \frac{\hat{H}}{H})^2(1 + \log \frac{H}{h})^2$. We can use the conjugate gradient method to obtain an estimate for the largest eigenvalue at the beginning of the computation to choose a proper u.

Let $\alpha = \frac{2}{l+u}$, $\mu = \frac{u+l}{u-l}$ and $Q = I - \alpha \left(\widehat{R}_{\widehat{D},\widehat{\Gamma}}^T \widetilde{T}^{-1} \widehat{R}_{\widehat{D},\widehat{\Gamma}}\right)\left(\widehat{R}_{\widehat{\Gamma}}^T \widetilde{T} \widehat{R}_{\widehat{\Gamma}}\right)$. Denote by σ_j the eigenvalues of Q.

If we choose u such that $\lambda_j < l + u$, we find that $1 - \frac{\cosh\left(k \cosh^{-1}(\mu\sigma_j)\right)}{\cosh\left(k \cosh^{-1}(\mu)\right)} > 0$, and we then have the following lemmas.

Lemma 6. *Given any $\mathbf{u_\Gamma} \in \widehat{\mathbf{W}}_\Gamma$, let $\mathbf{\Psi} = \Phi^T D_\Gamma R_\Gamma \mathbf{u}_\Gamma$ and select u such that $\lambda_j < u + l$. There then exist two functions $C_1(k)$ and $C_2(k)$ that*

$$C_1(k) \mathbf{\Psi}^T S_\Pi^{-1} \mathbf{\Psi} \leq \mathbf{\Psi}^T \widehat{S}_\Pi^{-1} \mathbf{\Psi}^T \leq C_2(k) \mathbf{\Psi}^T S_\Pi^{-1} \mathbf{\Psi},$$

where $C_1(k)$ and $C_2(k)$ are the minimum and maximum values, over all j, of $\left(1 - \frac{\cosh(k \cosh^{-1}(\mu\sigma_j))}{\cosh(k \cosh^{-1}(\mu))}\right)$.

Lemma 7. *Given any $\mathbf{u_\Gamma} \in \widehat{\mathbf{W}}_\Gamma$,*

$$C_1(k) \mathbf{u}_\Gamma^T M^{-1} \mathbf{u}_\Gamma \leq \mathbf{u}_\Gamma^T \widehat{M}^{-1} \mathbf{u}_\Gamma \leq C_2(k) \mathbf{u}_\Gamma^T M^{-1} \mathbf{u}_\Gamma,$$

where $C_1(k)$ and $C_2(k)$ are defined in Lemma 6.

We finally have

Theorem 2. *The condition number of the preconditioned operator using the three-level preconditioner $\widehat{M}^{-1}$ is bounded by $C \frac{C_2(k)}{C_1(k)}(1 + \log \frac{H}{h})^2$, where $C_1(k)$ and $C_2(k)$ are defined in Lemma 6 and $\frac{C_2(k)}{C_1(k)} \to 1$ as $k \to \infty$.*

5 Numerical experiments

We have applied our two three-level BDDC algorithms to the model problem (1). Here we only give results for two dimensions. We decompose the unit square into $\widehat{N} \times \widehat{N}$ subregions and each subregion into $N \times N$ subdomains with the side-length $\hat{H} = 1/\widehat{N}$ and $H = \hat{H}/N$, respectively. Equation (1) is discretized, in each subdomain, by conforming piecewise linear elements with

Table 1. Eigenvalue bounds and iteration counts with the preconditioner $\widetilde{M}^{-1}$

	$\frac{\hat{H}}{H}=4$ $\frac{H}{h}=4$			$\frac{H}{h}=4$ 4×4 subregions			$\frac{\hat{H}}{H}=4$ 4×4 subregions		
Subreg.	Iter.	Cond. #	$\frac{\hat{H}}{H}$	Iter.	Cond. #	$\frac{H}{h}$	Iter.	Cond. #
4×4	11	1.8096	4	11	1.8096	4	11	1.8096
8×8	11	1.8145	8	12	1.8536	8	14	2.4934
12×12	12	1.8159	12	12	1.8742	12	16	2.9758
16×16	12	1.8162	16	12	1.8912	16	17	3.3473
20×20	12	1.8164	20	12	1.9062	20	18	3.6546

Table 2. Eigenvalue bounds and iteration counts with the preconditioner $\widehat{M}^{-1}$, 4×4 subregions, $\frac{\hat{H}}{H}=16$ and $\frac{H}{h}=4$

		$u=3.2$				$u=6$				
k	Iter.	$C_1(k)$	λ_{min}	λ_{max}	Cond. #	Iter.	$C_1(k)$	λ_{min}	λ_{max}	Cond. #
1	20	0.4762	0.4829	2.7110	5.6141	24	0.2857	0.2899	1.8287	6.3086
2	13	0.8410	0.8540	1.8820	2.2038	16	0.6575	0.6670	2.3435	3.5134
3	11	0.9548	0.9981	1.9061	1.9098	12	0.8524	0.9286	1.9628	3.1136
4	11	0.9872	1.0019	1.8663	1.8629	12	0.9377	0.9795	1.9850	2.0266
5	11	0.9964	1.0006	1.8551	1.8541	12	0.9738	0.9983	1.9403	1.9437

a finite element diameter h. The preconditioned conjugate gradient iteration is stopped when the norm of the residual has been reduced by a factor of 10^{-8}.

We have carried out two different sets of experiments. All the experimental results are fully consistent with our theory. In the first set of experiments, we use the first preconditioner $\widetilde{M}^{-1}$ and take the coefficient $\rho = 1$ in half of the subregions and $\rho = 101$ in the neighboring subregions in a checkerboard pattern. Table 1 gives the iteration counts and condition number estimates with a change of the number of subregions, the number of subdomains, and the size of the subdomain problems.

In the second set of experiments, we use the second preconditioner $\widehat{M}^{-1}$ and take the coefficient $\rho \equiv 1$. We use the PCG to estimate the largest eigenvalue of $\left(\widehat{R}^T_{\widehat{D},\widehat{\Gamma}}\widetilde{T}^{-1}\widehat{R}_{\widehat{D},\widehat{\Gamma}}\right)\left(\widehat{R}^T_{\widehat{\Gamma}}\widetilde{T}\widehat{R}_{\widehat{\Gamma}}\right)$ which is approximately 3.2867. For 64×64 subdomains and $\frac{H}{h}=4$, we have a condition number estimate of 1.8380 for the two-level BDDC. We then select different values of u, the upper bound estimate of the eigenvalues for the preconditioned system, and k to see how the condition number changes. We also evaluate $C_1(k)$ for $k = 1, 2, 3, 4, 5$. From Table 2, we find that the smallest eigenvalue is bounded from below by $C_1(k)$ and that the condition number estimate approaches 1.8380, the value in the two-level case, as k increases. From these results, we see that if we can obtain precise estimate for the largest eigenvalue of $\left(\widehat{R}^T_{\widehat{D},\widehat{\Gamma}}\widetilde{T}^{-1}\widehat{R}_{\widehat{D},\widehat{\Gamma}}\right)\left(\widehat{R}^T_{\widehat{\Gamma}}\widetilde{T}\widehat{R}_{\widehat{\Gamma}}\right)$, we

need fewer Chebyshev iterations to obtain a condition number, similar to that of the two-level case. However, the iteration count is not very sensitive to the choice of u.

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