The primal alternatives of the FETI methods equipped with the lumped preconditioner

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In the past years, Domain Decomposition Methods (DDM) emerged as advanced solvers in several areas of Computational Mechanics. In particular, during the last decade, in the area of Solid and Structural Mechanics, they reached a considerable level of advancement and were shown to be more efficient than popular solvers, like advanced sparse direct solvers. The present contribution follows the lines of a series of recent publications by the authors on DDM. In these papers, the authors developed a unified theory of primal and dual methods and presented a family of DDM that were shown to be more efficient than previous methods. The present paper extends this work, presenting a new family of related DDM, thus enriching the theory of the relations between primal and dual methods.

1 Introduction

In the last decade Domain Decomposition Methods (DDM) have undergone a significant progress leading to a large number of methods and techniques, capable of giving solution to various problems of Computational Mechanics. In the field of Solid And Structural Mechanics, in particular, this fruitfull period led to the extensive parallel development of two large families of methods: (a) the Finite Element Tearing and Interconnecting (FETI) methods and (b) the Balancing Domain Decomposition (BDD) methods. Both introduced at the beginning of the 90s [FR91,Man93], these two categories of methods today include a large number of variants. However, their distinct theories led to the lack of extensive studies to interconnect them in the past. Thus, in the present decade two studies [KW01,FP03] attempted to determine the relations between the two methods.

In particular, the studies [FP03,FP04] set the basis of a unified theory of primal and dual DDM. This effort also led to the introduction of a new family of methods, under the name "Primal class of FETI methods", or in abbreviation "P-FETI methods". These methods are derived from the Dirichlet preconditioned FETI methods. They, thus, inherit the high computational efficiency properties of these methods, while their primal flavour gives them increased efficiency and robustness in ill-conditioned problems. However, so far there has not been presented a primal alternative for the lumped preconditioned FETI methods. Filling this hole is the object of the present study and even though the new formulations do not appear to share the same advantages as the P-FETI formulations, they serve the purpose of diversifying our knowledge of the relations of primal and dual methods.

This paper, thus, presents the primal alternatives of the lumped preconditioned FETI methods and is organised as follows: Section 2 presents the base formulation of the introduced methods and section 3 transforms the algorithms in a more economical form. Section 4 presents numerical results for comparing the new formulation with previous ones and section 5 gives some concluding statements.

2 Basic formulation of the primal alternatives of the FETI methods equipped with the lumped preconditioner

The P-FETI methods were built on the concept of preconditioning the Schur complement method with the first estimate of displacements obtained during the FETI methods. Accordingly, the primal counterparts of the lumped preconditioned methods will be obtained by similarly preconditioning the intact global problem. Thus, the following equation

$$Ku = f \Leftrightarrow L^T K^s L u = L^T f^s \tag{1}$$

will be preconditioned with the first displacement estimate of a FETI method. In eq. (1), K, u, and f represent the global stiffness matrix, displacement and force vectors, respectively, while

$$K^s = \begin{bmatrix} K^{(1)} & & \\ & \ddots & \\ & & K^{(n_s)} \end{bmatrix} \quad , \quad u^s = \begin{bmatrix} u^{(1)} \\ \vdots \\ u^{(n_s)} \end{bmatrix} \quad , \quad f^s = \begin{bmatrix} f^{(1)} \\ \vdots \\ f^{(n_s)} \end{bmatrix} \tag{2}$$

are the matrix block-diagonal assemblage of the correponding quantities of subdomains $s = 1, ..., n_s$ and L is a Boolean restriction matrix, such that $u^s = Lu$. Using the original FETI formulation, usually refered to as "one-level FETI" or "FETI-1", the following preconditioner for (1) is derived (this equation is obtained following an analysis almost identical to [FP03, section 6]):

$$\tilde{A}^{-1} = L_p^T \tilde{A}^{s^{-1}} L_p \tag{3}$$

where:

$$\tilde{A}^{s^{-1}} = H^T K^{s^+} H \quad , \quad H = I - B^T QG(G^T QG)^{-1} R^{s^T} \quad , \quad G = BR^s \tag{4}$$

Here, R^s and K^{s^+} are the block-diagonal assemblage of subdomain zero energy modes and generalized inverses of subdomain stiffness matrices, respectively, B is a mapping matrix such that $null(B) = range(L)$, Q is a symmetric positive definite matrix used in the FETI-1 coarse projector (see for instance [BDF$^+$00]), while L_p and B_p are scaled variants of L and B (see the expressions gathered from various DDM papers in [FP03]). Similar ideas lead to the corresponding preconditioners that are derived from other FETI variants. Comparing the lumped preconditioned FETI-1 method with the method of this section, it is noted that the present method has a significantly higher computational cost, because it operates on the full displacement vector u of the structure and also needs multiplications with the full stiffness matrices of the subdomains. In order to diminish its cost, this algorithm will be transformed into a more economical version, by respresenting its primal variables with dual variables.

3 Change of variables

The primal variables of the algorithm of the previous section will be represented with dual variables, based on the theorem: If the initial solution vector of the PCG algorithm applied for the solution of eq. (1) , with the preconditioner of eq. (3), is set equal to (In the following of this section we use the notation and steps of Algorithm 1):

$$u^0 = \tilde{A}^{-1} f \tag{5}$$

then there exist suitable vectors (denoted below with the subscript “1”), such that the following variables of the PCG can be written in the forms ($k = 0, 1, ...$):

$$z^k = -L_p^T \tilde{A}^{s^{-1}} B^T z_1^k \quad , \quad p^k = -L_p^T \tilde{A}^{s^{-1}} B^T p_1^k \tag{6}$$

$$r^k = L^T K^s B_p^T r_1^k \quad , \quad q^k = L^T K^s B_p^T q_1^k \tag{7}$$

Eqs. (6) - (7) allow expressing the PCG vectors, which have the size of the total number of degrees of freedom (d.o.f.), with respect to vectors whose size is equal to the row size of matrix B (which in turn is equal to the number of Lagrange multipliers used in dual DDM). They thus allow reducing the cost of the algorithm. The relatively small length of the present paper does not allow a full description of the proof for the above theorem. This proof is obtained by following the steps of the PCG and thus proving recursively the eqs. (6) - (7) (The full proof can be found in a larger version of this paper [FP05]). Using eqs. (6) - (7) and the definitions:

- Initialize

$$r^0 = f - Ku^0 \quad , \quad z^0 = \tilde{A}^{-1} r^0 \quad , \quad p^0 = z^0 \quad , \quad q^0 = Kp^0 \quad , \quad \eta^0 = \frac{{p^0}^T r^0}{{p^0}^T q^0}$$

- Iterate $k = 1, 2, ...$ until convergence

$$u^k = u^{k-1} + \eta^{k-1} p^{k-1} \quad , \quad r^k = r^{k-1} - \eta^{k-1} q^{k-1} \quad , \quad z^k = \tilde{A}^{-1} r^k$$
$$p^k = z^k - \sum_{i=0}^{k-1} \frac{{z^k}^T q^i}{{p^i}^T q^i} p^i \quad , \quad q^k = Kp^k \quad , \quad \eta^k = \frac{{p^k}^T r^k}{{p^k}^T q^k}$$

Algorithm 1. The PCG algorithm for solving system $Ku = f$ preconditioned with $\tilde{A}^{-1}$ (full reorthogonalization)

$$z_2^k = B{\tilde{A}^s}^{-1} B^T z_1^k \quad , \quad z_3^k = B_p K^s B_p^T z_2^k \tag{8}$$

$$p_2^k = B{\tilde{A}^s}^{-1} B^T p_1^k \quad , \quad p_3^k = B_p K^s B_p^T p_2^k \tag{9}$$

$$r_2^k = B_p K^s B_p^T r_1^k \quad , \quad r_3^k = B{\tilde{A}^s}^{-1} B^T r_2^k \tag{10}$$

$$q_2^k = B_p K^s B_p^T q_1^k \quad , \quad q_3^k = B{\tilde{A}^s}^{-1} B^T q_2^k \tag{11}$$

it is thus shown following the proof of the above theorem that the PCG algorithm for solving eq. (1) with preconditioner of eq. (3) is transformed into Algorithm 2 (in the case of full reorthogonalization). In Algorithm 2, it is worth noting that even though the formulation is primal, the final algrorithm is very similar to the algorithm of the FETI-1 method with the lumped preconditioner. In particular:

- The matrices $B{\tilde{A}^s}^{-1} B^T$ and $B_{p_b}^T K_{bb}^s B_{p_b}^T$ that are used during the iterations are equal to the FETI-1 matrix operator and lumped preconditioner, respectively.
- The algorithm iterates on vectors of the size of the Lagrange multipliers.
- The residual vanishes in internal d.o.f. of the subdomains, when these d.o.f. are not adjucent to the interface, again as in FETI-1 with the lumped preconditioner.

On the other hand, each iteration of the present algorithm requires more linear combinations of vectors than a dual algorithm. These operations become important in the case of reorthogonalization. In this case, the required dot products ${z_1^k}^T (q_3^i - q_1^i)$, $i = 0, ..., k-1$ imply the same computational cost as in FETI-1, because at each iteration $q_3^k - q_1^k$ is computed and stored. However, compared to FETI-1, this algorithm requires twice as many linear combinations for computing the vectors p_1^k and p_2^k, that represent the direction vectors p^k. In total, in this algorithm reorthogonalization requires 50% more floating point operations than in FETI-1. In addition, while FETI-1 reorthogonalization requires storing two vectors per iteration, here it is required to store the three vectors p_1^k, p_2^k and $q_3^k - q_1^k$, which implies 50% higher memory requirements for reorthogonalization in Algorithm 2.

- Initialize

$$u^0 = L_p^T \tilde{A}^{s^{-1}} L_p f \quad , \quad \tilde{u}^0 = 0 \quad , \quad r_1^0 = B\tilde{A}^{s^{-1}} L_p f$$
$$r^0 = \begin{bmatrix} L_b^T K_{bb}^s \\ K_{ib}^s \end{bmatrix} B_{p_b}^T r_1^0 \quad , \quad p_1^0 = z_1^0 = B_{p_b}^T K_{bb}^s B_{p_b}^T r_1^0$$
$$q_1^0 = p_2^0 = r_3^0 = z_2^0 = B\tilde{A}^{s^{-1}} B^T z_1^0 \quad , \quad q^0 = \begin{bmatrix} L_b^T K_{bb}^s \\ K_{ib}^s \end{bmatrix} B_{p_b}^T q_1^0$$
$$p_3^0 = q_2^0 = B_{p_b}^T K_{bb}^s B_{p_b}^T q_1^0 \quad , \quad \eta^0 = \frac{(p_3^{0^T} - p_1^{0^T}) r_1^0}{(p_3^{0^T} - p_1^{0^T}) q_1^0}$$

- Iterate $k = 1, 2, ...$ until convergence ($\left\|r^k\right\| < \varepsilon$)

$$\tilde{u}_1^k = \tilde{u}_1^k + \eta^{k-1} p_1^{k-1} \quad , \quad r^k = r^{k-1} - \eta^{k-1} q^{k-1} \quad , \quad r_1^k = r_1^{k-1} - \eta^{k-1} q_1^{k-1}$$
$$z_1^k = r_2^k = r_2^{k-1} - \eta^{k-1} q_2^{k-1} \quad , \quad r_3^k = z_2^k = B\tilde{A}^{s^{-1}} B^T z_1^k$$
$$q_3^{k-1} = \left(1/\eta^{k-1}\right)\left(r_3^{k-1} - r_3^k\right) \quad , \quad p_1^k = z_1^k - \sum_{i=0}^{k-1} \frac{z_1^{k^T}(q_3^i - q_1^i)}{p_1^{i^T}(q_3^i - q_1^i)} p_1^i$$
$$q_1^k = p_2^k = z_2^k - \sum_{i=0}^{k-1} \frac{z_1^{k^T}(q_3^i - q_1^i)}{p_1^{i^T}(q_3^i - q_1^i)} p_2^i \quad , \quad q^k = \begin{bmatrix} L_b^T K_{bb}^s \\ K_{ib}^s \end{bmatrix} B_{p_b}^T p_2^k$$
$$p_3^k = q_2^k = B_{p_b}^T K_{bb}^s B_{p_b}^T p_2^k \quad , \quad \eta^k = \frac{(p_3^{k^T} - p_1^{k^T}) r_1^k}{(p_3^{k^T} - p_1^{k^T}) q_1^k}$$

- After convergence

$$u^k = u^0 - L_p^T \tilde{A}^{s^{-1}} B^T \tilde{u}_1^k$$

Algorithm 2: The primal alternative of the FETI-1 method with the lumped preconditioner (full reorthogonalization)

4 Numerical results

We have implemented the FETI-1 and FETI-DP methods with the lumped preconditioner and their primal alternatives in our Matlab code and we consider the 3-D elasticity problem of Fig. 1. This cubic structure is composed of five layers of two different materials and is discretized with $28 \times 28 \times 28$ 8-node brick elements. Additionally, it is pinned at the four corners of its left surface. Various ratios E_A/E_B of the Young modulus and ρ_A/ρ_B of the density of the two materials are considered in the paper, while their Poisson ratio is set equal to $\nu_A = \nu_B = 0.30$. Two decompositions P1 and P2 of this heterogeneous model of $73,155$ d.o.f. in 100 subdomains, are considered (see [FP03] for details).

Table 1 presents the iterations required by primal and dual formulations of the lumped preconditioned FETI-1 method. The results show that like in the case of comparing dual and primal formulations of the Dirichlet preconditioned FETI methods, the iterations of the two formulations of the lumped preconditioned FETI-1 methods are comparable. More precisely, it is noted that in the more ill-conditioned cases the primal method performs slightly less iterations (up to 11%) than the dual one. In fact, judging also from many other tests that we have performed comparing the two formulations of FETI-1 and FETI-DP with the lumped preconditioner, it appears that the difference between the number of iterations of primal and dual formulations in ill-conditioned problems is more pronounced in the case of the lumped preconditioner than in the case of the Dirichlet preconditioner. A probable explanation is that the lumped preconditioned methods lead by themselves to more ill-conditioned systems than the Dirichlet ones.

On the other hand, bearing in mind that the primal formulation implies a 50% higher reorthogonalization cost, we conclude that statistically the primal formulation will be probably slower than the dual one in well-conditioned problems and probably faster in ill-conditioned problems with relatively low reorthogonalization cost. In addition, in the case of the lumped preconditioner, our results do not show the increased robustness (measured in terms of the maximum achievable solution accuracy in ill-conditioned problems) of the primal formulation that has been seen in the case of the P-FETI formulations. A probable explanation of this observation is given by the increased operations required in each iteration of the primal algorithm as oposed to the dual one and also by the fact that due to setting the initial solution vector equal to eq. (20), the initial residual of the primal methods is equal to the initial residual of the dual methods (see the expression of the residual r^0 in Algorithm 2, which is equal to the initial residual of the FETI-1 method). Thus, contrary to the P-FETI formulations, the residuals of the primal formulations of the lumped preconditioned FETI methods begin from relatively high values, as in the dual formulations.

5 Conclusions

The roots of the presented in this paper work can be traced back to the paper [FP03]. This paper introduced the P-FETI methods, as the primal alternatives of the Dirichlet preconditioned FETI methods. Compared to the original FETI formulations, the P-FETI methods present the advantage of being more robust and faster in the solution of ill-conditioned problems. [FP03] also introduced an open question of the existence or not of a primal alternative for the lumped preconditioned FETI methods. In the last years it has become clear that the the lumped preconditioner leads to faster solutions, in the cases where a problem needs to be decomposed in a relatively small number of subdomains. This case and also the case where the lumped precondtioner leads to less memory consumption (in large problems where memory consumption can be the main issue), appear to be the uses of the lumped preconditioner in modern DDM practice.

The present work introduces the primal alternatives of the lumped preconditioned FETI methods. These new formulations do not appear to present the advantages of the P-FETI formulations, since they are slightly slower or faster

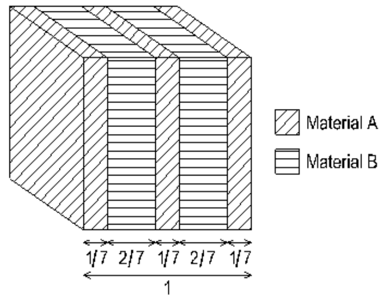

Fig. 1. A cubic structure composed of two materials

Table 1. Number of iterations (Tolerance:10^{-3}) of the lumped Preconditioned FETI-1 method and its primal alternative for the solution of the example of Fig. 1

Ratio of Young moduli	Type of decomposition	Dual formulation	Primal formulation
10^0	P1	25	24
10^3	P1	44	41
10^3	P2	25	24
10^6	P1	30	26
10^6	P2	53	47

than their dual counterparts depending on the problem and do not exhibit higher robustness properties than the dual methods. Their principal value lies in the fact that they add a new level of completion to the theory of the relations of primal and dual methods. The fact that a primal algorithm can be turned to an algorithm which uses dual operators and vectors appears to be new in DDM literature. It is also worth noting that the same transformations used in this paper can be used in the P-FETI and the BDD methods in order to transform them into algorithms that operate on dual quantities. This and many other recent studies [KW01,MDT03] show more and more that primal and dual formulations are closely connected.

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