Thirteenth International Conference on Domain Decomposition Methods
Editors: N. Debit, M.Garbey, R. Hoppe, J. Périaux, D. Keyes, Y. Kuznetsov

11 Rate of Convergence for Parallel Subspace Correction Methods for nonlinear variational inequalities

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Introduction

Given a reflexive Banach space V and a convex functional $F : V \mapsto R$, we shall consider the following nonlinear optimization problem

$$\min_{v \in K} F(v), \quad K \subset V \,. \tag{1}$$

The nonempty convex subset K is assumed to be closed in the strong topology of V. We are interested in the case that the space V can be decomposed into a sum of subspaces V_i, i.e.

$$V = V_1 + V_2 + \cdots + V_m = \sum_{i=1}^{m} V_i \,. \tag{2}$$

This means that for any v, there exists $v_i \in V_i$ such that $v = \sum_{i=1}^{m} v_i$.

After the decomposition of the space as in (2), there are two different ways to solve the nonlinear problem (1). The first alternative is to decompose K into a sum of $K_i \subset V_i,\ i = 1, 2, \cdots, m$, i.e.

$$K = K_1 + K_2 + \cdots + K_m = \sum_{i=1}^{m} K_i,$$

and then solve a minimization problem over each subset K_i in parallel or sequentially. The convergence analysis and numerical experiments have been done in [Tai00].

For the second alternative, we only need to decompose the space V as in (2), but we do not need to decompose the constraint set K, see the next section for the detailed algorithms. Uniform linear convergence rate analysis for these algorithms is still missing in the literature. The contribution of this work to give a mesh independent linear convergence rate estimate for domain decomposition and multigrid methods for these algorithms. The techniques used in the analysis are extensions of the techniques used in [TE98, Tai00, TT98, TX01].

We will find the proper assumptions on the decomposed subspaces to guarantee that the algorithms will have a uniform linear convergence rate and then we verify that these assumptions are really valid for domain decomposition and multigrid methods.

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The algorithms and some assumptions

Algorithm 1 For a given $u^n \in K$ and $\alpha \in (0, 1/m)$, compute $e_i^{n+1} \in V_i$ in parallel for $i = 1, 2, \cdots, m$ such that

$$e_i^{n+1} = \arg \min_{\substack{v_i + u^n \in K \\ v_i \in V_i}} G(v_i) \quad \text{with} \quad G(v_i) = F(u^n + v_i). \tag{3}$$

and then update

$$u^{n+1} := u^n + \alpha \sum_{i=1}^{m} e_i^{n+1}$$

■

Algorithm 2 For a given $u^n \in K$ and $\alpha \in (0, 1)$, compute $e_i^{n+1} \in V_i$ sequentially for $i = 1, 2, \cdots, m$ such that

$$e_i^{n+1} = \arg \min_{\substack{v_i + u^{n+\frac{i-1}{m}} \in K \\ v_i \in V_i}} G(v_i) \quad \text{with} \quad G(v_i) = F(u^{n+\frac{i-1}{m}} + v_i). \tag{4}$$

and update

$$u^{n+\frac{i}{m}} := u^{n+\frac{i-1}{m}} + \alpha e_i^{n+1}$$

■

For the minimization functional F, we only need to assume that F is Gâteaux differentiable (see [ET76]) and that there exists a constant $\kappa > 0$ such that

$$\langle F'(w) - F'(v), w - v \rangle \geq \kappa \|w - v\|_V^2, \quad \forall w, v \in V \,. \tag{5}$$

Here $\langle \cdot, \cdot \rangle$ is the duality pairing between V and its dual space V', i.e. the value of a linear function at an element of V. Under the assumption (5), problem (1) has a unique solution, see [ET76, p. 35]. For some nonlinear problems, the constant κ may depend on v and w.

As in [TE98, TT98, TX01], we shall use two constants in the estimation of the rate of the convergence of the algorithms. First, we assume that there exists a constant $C_1 > 0$ and this constant is only related to the decomposition (2). With the constant C_1 and the decomposition (2), it is assumed that for any $v, w \in K$, one can find $z_i \in V_i$ to satisfy

$$v - w = \sum_{i=1}^{m} z_i, \quad z_i + w \in K, \quad \text{and} \quad \left(\sum_{i=1}^{m} \|z_i\|_V^2 \right)^{\frac{1}{2}} \leq C_1 \|v - w\|_V \,. \tag{6}$$

In addition to the assumption of the existence of such a constant C_1, we also assume that there is a $C_2 > 0$ which is the least constant satisfying the following inequality for any $w_{ij} \in V, u_i \in V_i$ and $v_j \in V_j$:

$$\sum_{i,j=1}^{m} \left| \langle F'(w_{ij} + u_i) - F'(w_{ij}), v_j \rangle \right| \leq C_2 \left(\sum_{i=1}^{m} \|u_i\|_V^2 \right)^{\frac{1}{2}} \left(\sum_{j=1}^{m} \|v_j\|_V^2 \right)^{\frac{1}{2}}. \tag{7}$$

Later we shall show that these assumptions are valid for domain decomposition and multigrid methods. Moreover, the constants C_1 and C_2 are mesh independent.

The convergence of the parallel subspace correction method

We shall only do the convergence analysis for Algorithm 1. For notation simplicity, we define u to be the unique solution of (1) and for any $n \geq 0$ we define

$$u^n = \sum_{i=1}^{m} u_i^n, \;\; \hat{u}^{n+1} = u^n + \sum_{i=1}^{m} e_i^{n+1}, \; d_n = F(u^n) - F(u). \tag{8}$$

Theorem 1 *Assuming that the space decomposition satisfies (6), (7) and that the functional F satisfies (5). Then for Algorithms 1, we have*

$$\frac{F(u^{n+1}) - F(u)}{F(u^n) - F(u)} \leq 1 - \frac{\alpha}{(\sqrt{1 + C^*} + \sqrt{C^*})^2}, \tag{9}$$

with

$$C^* = \left(C_2 + \frac{(C_1 C_2)^2}{2\kappa} \right) \frac{2}{\kappa}. \tag{10}$$

Proof. Since e_i^{n+1} minimizes (3), it satisfies (see [ET76])

$$\langle F'(u^n + e_i^{n+1}), v_i - e_i^{n+1} \rangle \geq 0, \quad \forall v_i \in V_i \quad \text{satisfying} \quad v_i + u^n \in K. \tag{11}$$

Under the assumption of (5), it is known that (see [TE98, Lemma 3.2])

$$F(w) - F(v) \geq \langle F'(v), w - v \rangle + \frac{\kappa}{2} \|w - v\|_V^2 \;, \quad \forall v, w \in V. \tag{12}$$

Using these results, we get that

$$\begin{aligned} F(u^n) - F(u^{n+1}) &= F(u^n) - F\left(u^n + \alpha \sum_{i=1}^{m} e_i^{n+1} \right) \\ &= F(u^n) - F\left(\sum_{i=1}^{m} \alpha (u^n + e_i^{n+1}) + (1 - \alpha m) u^n \right) \\ &\geq F(u^n) - \alpha \sum_{i=1}^{m} F(u^n + e_i^{n+1}) - (1 - \alpha m) F(u^n) \\ &= \alpha \sum_{i=1}^{m} (F(u^n) - F(u^n + e_i^{n+1})) \\ &\geq \frac{\alpha \kappa}{2} \sum_{i=1}^{m} \|e_i^{n+1}\|_V^2 \qquad \text{(using (11) and (12)).} \end{aligned} \tag{13}$$

The argument used to get the above estimates is the same as the unconstrained case, see [TX01]. For notational simplicity, we introduce for a given i

$$\sigma_j^n = \begin{cases} u^n + \displaystyle\sum_{k=i}^{j+i-1} e_k^{n+1} \;, & \forall j \in [1, m - i + 1] \;; \\ u^n + \displaystyle\sum_{k=i}^{m} e_k^{n+1} + \sum_{k=1}^{j-m+i-1} e_k^{n+1} \;, & \forall j \in [m - i + 2, m] \;. \end{cases} \tag{14}$$

It is clear that σ_j^n depends on i. Moreover, we see that

$$\begin{aligned}
\sigma_1^n &= u^n + e_i^{n+1}, \\
\sigma_2^n &= u^n + e_i^{n+1} + e_{i+1}^{n+1}, \\
&\vdots \\
\sigma_m^n &= u^n + \sum_{k=1}^m e_k^{n+1}.
\end{aligned}$$

It is easy to see that

$$F'\Big(u^n + \sum_{j=1}^m e_j^{n+1}\Big) - F'(u^n + e_i^{n+1}) = \sum_{j=2}^m \big(F'(\sigma_j^n) - F'(\sigma_{j-1}^n)\big) \ . \tag{15}$$

From assumption (6), there exists $z_i^n \in V_i$ such that

$$u - u^n = \sum_{i=1}^m z_i^n, \qquad z_i^n + u^n \in K, \qquad \Big(\sum_{i=1}^m \|z_i^n\|^2\Big)^{\frac{1}{2}} \le C_1 \|u - u^n\|. \tag{16}$$

We shall now use all of the above to estimate

$$\begin{aligned}
&\langle F'(\hat{u}^{n+1}), \hat{u}^{n+1} - u\rangle = \sum_{i=1}^m \Big\langle F'(\hat{u}^{n+1}), e_i^{n+1} - z_i^n \Big\rangle \\
&\le \sum_{i=1}^m \Big\langle F'(\hat{u}^{n+1}) - F'(u + e_i^{n+1}), e_i^{n+1} - z_i^n \Big\rangle \qquad \text{(using (11) and (16))} \\
&= \sum_{i=1}^m \sum_{j=2}^m \Big\langle F'(\sigma_j^n) - F'(\sigma_{j-1}^n), e_i^{n+1} - z_i^n \Big\rangle \qquad \text{(using (15))} \\
&\le C_2 \Big(\sum_{i=1}^m \|e_i^{n+1}\|^2\Big)^{\frac{1}{2}} \Big(\sum_{i=1}^m \|e_i^{n+1} - z_i^n\|^2\Big)^{\frac{1}{2}} \qquad \text{(using (7))} \qquad (17) \\
&\le C_2 \left(\sum_{i=1}^m \|e_i^{n+1}\|^2\right)^{\frac{1}{2}} \left(\left(\sum_{i=1}^m \|e_i^{n+1}\|^2\right)^{\frac{1}{2}} + C_1 \|u - u^n\|\right) \text{(using (6), (8) and (16))} \\
&= C_2 \sum_{i=1}^m \|e_i^{n+1}\|^2 + C_1 C_2 \Big(\sum_{i=1}^m \|e_i^{n+1}\|^2\Big)^{\frac{1}{2}} \|u - u^n\|.
\end{aligned}$$

The rest of the proof is the same as in [Tai00]. ∎

The general theory developed for (1) will be applied to the following obstacle problem in connection with finite element approximations:

$$\text{Find } u \in K, \quad \text{such that} \quad a(u, v - u) \ge f(v - u), \quad \forall v \in K, \tag{18}$$

with $a(v, w) = \int_\Omega \nabla v \cdot \nabla w \, dx$, $K = \{v \in H_0^1(\Omega) | \, v(x) \ge \psi(x) \text{ a.e. in } \Omega\}$. For the analysis, it can be assumed without loss of any generality that

$$\psi = 0. \tag{19}$$

It is well known that the above problem is equivalent to the following minimization problem

$$\min_{v \in K} F(v), \quad F(v) = \frac{1}{2} a(v,v) - f(v), \tag{20}$$

assuming that $f(v)$ is a linear functional on $H_0^1(\Omega)$. For the obstacle problem (18), the minimization space $V = H_0^1(\Omega)$. Correspondingly, we have $\kappa = 1$ for assumption (5).

Overlapping domain decomposition

In this section we apply our algorithms to the overlapping domain decomposition method. For the domain Ω, we first partition it into a coarse mesh division $\{\mathcal{T}_H\}$ with a mesh size H and then refine it into a fine mesh partition $\{\mathcal{T}_h\}$ with a mesh size $h < H$. We assume that both the coarse mesh and the fine mesh are shape-regular. Let $\{\Omega_i\}_{i=1}^M$ be a nonoverlapping domain decomposition for Ω and each Ω_i is the union of some coarse mesh elements. Let $S^H \subset W^{1,\infty}(\Omega)$ and $S^h \subset W^{1,\infty}(\Omega)$ be the continuous, piecewise linear finite element spaces over the H-level and h-level subdivisions of Ω respectively. More specifically,

$$S^H = \left\{ v \in W^{1,\infty}(\Omega_H) \middle| \quad v|_{\Omega_i} \in P_1(\Omega_i), \forall i \right\},$$

$$S^h = \left\{ v \in W^{1,\infty}(\Omega_h) \middle| \quad v|_{\mathcal{T}} \in P_1(\mathcal{T}), \forall \mathcal{T} \in \mathcal{T}_h \right\}.$$

For each Ω_i, we consider an enlarged subdomain Ω_i^δ consisting of elements $\mathcal{T} \in \mathcal{T}_h$ with $\mathrm{dist}(\mathcal{T}, \Omega_i) \le \delta$. The union of Ω_i^δ covers $\bar{\Omega}_h$ with overlaps of size δ. Let us denote the piecewise linear finite element space with zero traces on the boundaries $\partial\Omega_i^\delta \backslash \partial\Omega$ as $S^h(\Omega_i^\delta)$. Then one can show that

$$S^h = \sum_{i=1}^{M} S^h(\Omega_i^\delta) \quad \text{and} \quad S^h = S^H + \sum_{i=1}^{M} S^h(\Omega_i^\delta). \tag{21}$$

For the overlapping subdomains, assume that there exist m colors such that each subdomain Ω_i^δ can be marked with one color, and the subdomains with the same color will not intersect with each other. For suitable overlaps, one can always choose $m = 2$ if $d = 1; m \le 4$ if $d = 2; m \le 8$ if $d = 3$. Let Ω_i^c be the union of the subdomains with the i^{th} color, and

$$V_i = \{v \in S^h | \quad v(x) = 0, \quad x \notin \Omega_i^c\} \; i = 1, 2, \cdots, m.$$

By denoting subspaces $V_0 = S_H, V = S_h$, we get from (21) that

$$a). \quad V = \sum_{i=1}^{m} V_i \qquad \text{and} \qquad b). \quad V = V_0 + \sum_{i=1}^{m} V_i. \tag{22}$$

Note that the summation index is from 0 to m instead of from 1 to m when the coarse mesh is added.

Let $\{\theta_i\}_{i=1}^m$ be a partition of unity with respect to $\{\Omega_i^c\}_{i=1}^m$, i.e. $\theta_i \in V_i$, $\theta_i \ge 0$ and $\sum_{i=1}^m \theta_i = 1$. It can be chosen so that

$$|\nabla\theta_i| \le C/\delta, \quad \theta_i(x) = \begin{cases} 1 & \text{if } x \in \tau, \text{ distance } (\tau, \partial\Omega_i^c) \ge \delta \text{ and } \tau \subset \Omega_i^c, \\ 0 & \text{on } \overline{\Omega \backslash \Omega_i^c}. \end{cases} \tag{23}$$

In the following, we shall give the definition of a nonlinear interpolation operator $I_H^{\ominus} : S_h \mapsto S_H$ which was introduced in [Tai00]. Denote by $\mathcal{N}_H = \{x_0^i\}_{i=1}^{n_0}$ all the interior nodes for $\mathcal{T}_H$. For a given x_0^i, let ω_i be the union of the mesh elements of $\mathcal{T}_H$ having x_0^i as one of its vertices, i.e. $\omega_i := \cup\{\tau \in \mathcal{T}_H, x_0^i \in \bar{\tau}\}$. Let $\{\phi_0^i\}_{i=1}^{n_0}$ be the associated nodal basis functions of S_H satisfying $\phi_0^i(x_0^k) = \delta_{ik}$, $\phi_0^i \geq 0$, $\forall i$ and $\sum_i \phi_0^i(x) = 1$. It is clear that ω_i is the support of ϕ_0^i. Given a nodal point $x_0^i \in \mathcal{N}_H$ and a $v \in S_h$, let $I_i v = \min_{\omega_i} v(x)$. The interpolated function is then defined as

$$I_H^{\ominus} v := \sum_{x_0^i \in \mathcal{N}_H} (I_i v)\phi_0^i(x).$$

From the definition, it is easy to see that

$$I_H^{\ominus} v \leq v, \quad \forall v \in S_h, \tag{24}$$

$$I_H^{\ominus} v \geq 0, \quad \forall v \geq 0, v \in S_h. \tag{25}$$

Moreover, the interpolation for a given $v \in S_h$ on a finer mesh is always bigger than the corresponding interpolation on a coarser mesh due to the fact that each coarser mesh element contains several finer mesh elements, i.e.

$$I_{h_1}^{\ominus} v \leq I_{h_2}^{\ominus} v, \quad \forall h_1 \geq h_2 \geq h, \quad \forall v \in S_h. \tag{26}$$

In addition, the interpolation operator also has the following approximation properties (c.f. [Tai00])

$$\|I_H^{\ominus} v - I_H^{\ominus} w - (v - w)\|_0 \leq c_d H |v - w|_1, \forall v, w \in S_h \tag{27}$$

$$\|I_H^{\ominus} v - v\|_0 \leq c_d H |v|_1, \qquad \|I_H^{\ominus} v - I_H^{\ominus} w\|_1 \leq c_d \|v - w\|_1, \forall v, w \in S_h, \tag{28}$$

where $c_d = C$ if $d = 1$; $c_d = C\left(1 + \left|\log \frac{H}{h}\right|^{\frac{1}{2}}\right)$ if $d = 2$ and $c_d = C\left(\frac{H}{h}\right)^{\frac{1}{2}}$ if $d = 3$.

We first use decomposition (22.a) to decompose the finite element space $V = S_h$, i.e. the coarse mesh is not used in the computations. Let I_h be the Lagrangian interpolation operator which uses the function values at the h-level nodes. In order to estimate constant C_1, we take $z_i = I_h(\theta_i(v - w))$ for any $v, w \in K$. As $\sum_{i=1}^m \theta_i = 1$, thus $\sum_{i=1}^m z_i = v - w$ using the linearity of I_h. Moreover,

$$z_i + w = I_h(\theta_i(v - w) + w).$$

Under assumption (19), it follows from the fact that $\theta_i \in (0, 1)$ and the convexity of K that $z_i \in K$. It is easy to prove that the following estimate is correct and the proof is exactly the same as for the linear unconstrained case [SBG96, Xu92]:

$$C_1 \leq C(1 + \delta^{-1}), \quad C_2 = \sqrt{m}.$$

If we shall use the coarse mesh, then the decomposition is as given in (22.b). The estimation for C_2 is the same as for linear problems, we just need to find the biggest constant which satisfies (6).

In order to show that assumption (6) is valid for decomposition (22.b), we first decompose $v - w$ into

$$v - w = \sigma^{\oplus} - \sigma^{\ominus}, \quad \sigma^{\oplus} = \max(0, v - w),\ \sigma^{\ominus} = \max(0, w - v), \tag{29}$$

and then define $z_0 \in V_0$ as

$$z_0 = I_H^{\ominus} I_h \sigma^{\oplus} - I_H^{\ominus} I_h \sigma^{\ominus}.$$

Under assumption (19), we see that $v, w \geq 0$. From (24) and (25), it is true that

$$0 \leq I_H^{\ominus} I_h \sigma^{\oplus} \leq I_h \sigma^{\oplus} \leq v, \quad 0 \leq I_H^{\ominus} I_h \sigma^{\ominus} \leq I_h \sigma^{\ominus} \leq w \text{ and so } -w \leq z_0 \leq v. \tag{30}$$

Due to the special structure of the functions $\sigma^{\oplus}$ and $\sigma^{\ominus}$, it is in fact easy to prove that

$$|I_h \sigma^{\oplus}|_1 \leq C|v-w|_1, \quad |I_h \sigma^{\ominus}|_1 \leq C|v-w|_1, \tag{31}$$

where the constant C only depends on the minimal angle conditions. From the above inequalities and estimate (27), it is easy to see that

$$\|z_0 - (v-w)\|_l \leq c_d H^{1-l} |v-w|_1, \quad l = 0, 1.$$

Taking

$$z_i = I_h(\theta_i(v - w - z_0)), \ i = 1, 2, \cdots, m,$$

we get by using the linearity of I_h, the equality $I_h w = w$ and (30) that

$$z_0 + \sum_{i=1}^{m} z_i = v - w, \quad z_0 + w \geq 0, \quad \text{and} \quad z_i + w = I_h(\theta_i(v - z_0) + (1-\theta_i)w) \geq 0.$$

Using the approximation properties (27)-(28), the following estimate is correct and the proof is the same as for the linear unconstrained case, see [TX01]:

$$\left(\|z_0\|_1^2 + \sum_{i=1}^{m} \|z_i\|_1^2 \right)^{\frac{1}{2}} \leq (m+1)^{\frac{1}{2}} c_d \left(1 + \left(\frac{H}{\delta} \right)^{\frac{1}{2}} \right) |v-w|_1. \tag{32}$$

Thus it is shown that assumption (6) is valid for decomposition (22.b) with

$$C_1 = (m+1)^{\frac{1}{2}} c_d \left(1 + \left(\frac{H}{\delta} \right)^{\frac{1}{2}} \right).$$

Assumption (7) has been shown to be correct for the decomposition (22.b) with $C_2 = \sqrt{m+1}$ and m being the number of colors, see [TX01], see also [SBG96, DW94, Xu92].

Multigrid decomposition

A multigrid algorithm is built upon the subspaces that are defined on a nested sequence of finite element partitions. We assume that the finite element partition $\mathcal{T}$ is constructed by a successive refinement process. More precisely, $\mathcal{T} = \mathcal{T}_J$ for some $J > 1$, and $\mathcal{T}j$ for, $j \leq J$ is a nested sequence of quasi-uniform finite element partitions, i.e. $\mathcal{T}j$ consist of finite elements $\mathcal{T}j = \{\tau_j^i\}$ of size h_j such that $\Omega = \cup_i \tau_j^i$ for which the quasi-uniformity constants are independent of j and τ_{j-1}^l is a union of elements of $\{\tau_j^i\}$. We further assume that there is a constant $\gamma < 1$, independent of j, such that h_j is proportional to γ^{2j}.

As an example, in the two dimensional case, a finer grid is obtained by connecting the midpoints of the edges of the triangles of the coarser grid, with $\mathcal{T}_1$ being the given coarsest initial triangulation, which is quasi-uniform. In this example, $\gamma = 1/\sqrt{2}$. We can use much smaller γ in constructing the meshes, but the constant C_1 is getting larger when γ is becoming smaller, see (35).

Corresponding to each finite element partition $\mathcal{T}j$, a finite element space $\mathcal{M}_j$ can be defined by

$$\mathcal{M}_j = \{v \in W^{1,\infty}(\Omega) : v|_\tau \in \mathcal{P}_1(\tau), \quad \forall\, \tau \in \mathcal{T}j\}.$$

Each finite element space $\mathcal{M}_j$ is associated with a nodal basis, denoted by $\{\phi_j^i\}_{i=1}^{n_j}$ satisfying

$$\phi_j^i(x_j^k) = \delta_{ik}$$

where $\{x_j^k\}_{k=1}^{n_j}$ is the set of all nodes of the elements of $\mathcal{T}_j$. Associated with each such a nodal basis function, we define a one dimensional subspace as follows

$$\mathcal{M}_j^i = \text{span}\,(\phi_j^i).$$

It is easy to see that

$$\mathcal{M}_J = \sum_{j=1}^{J} \sum_{i=1}^{n_j} \mathcal{M}_j^i. \tag{33}$$

Similar as for the two-level decomposition, we first decompose $v - w$ for any $v, w \ge 0$ as in (29). We then define $\sigma_j^{\oplus} = I_{h_j}^{\ominus} I_h \sigma^{\oplus}$, $\sigma_j^{\ominus} = I_{h_j}^{\ominus} I_h \sigma^{\ominus}$ for $j = 1, 2, \cdots, J$ and $\sigma_0^{\oplus} = 0$, $\sigma_0^{\ominus} = 0$. From properties (24)–(28) and the fact that $v, w \ge 0$, it is true that

$$0 \le \sigma_j^{\oplus} \le I_h \sigma^{\oplus} \le v, \quad \|\sigma_j^{\oplus} - \sigma^{\oplus}\|_l \le \tilde{c}_d h_j^{1-l} |v - w|_1,\ l = 0, 1.$$

$$0 \le \sigma_j^{\ominus} \le I_h \sigma^{\ominus} \le w, \quad \|\sigma_j^{\ominus} - \sigma^{\ominus}\|_l \le \tilde{c}_d h_j^{1-l} |v - w|_1,\ l = 0, 1,$$

where

$$\tilde{c}_d = \begin{cases} C, & \text{if} \quad d = 1; \\ C(1 + |\log h|^{\frac{1}{2}}), & \text{if} \quad d = 2; \\ Ch^{-\frac{1}{2}}, & \text{if} \quad d = 3. \end{cases}$$

Define

$$z_J = v - w - \sigma_{J-1}^{\oplus} + \sigma_{J-1}^{\ominus}, \quad z_j = \sigma_j^{\oplus} - \sigma_{j-1}^{\oplus} - (\sigma_j^{\ominus} - \sigma_{j-1}^{\ominus}),\ j = 1, 2, \cdots J - 1.$$

From (25) and (26), we see that $\sigma_j^{\oplus} - \sigma_{j-1}^{\oplus} \ge 0$ and $\sigma_{j-1}^{\ominus} \ge 0$, we thus get

$$z_j + w \ge w - \sigma_j^{\ominus} \ge 0, \quad j = 1, 2, \cdots, J - 1. \tag{34}$$

Similar argument shows that $z_J + w = v - \sigma_{J-1}^{\oplus} + \sigma_{J-1}^{\ominus} \ge 0$. The fact that $\sum_{j=1}^{J} z_j = v - w$ is an easy consequence of the definitions of z_j. A further decomposition of z_j is given by

$$z_j = \sum_{i=1}^{n_j} z_j^i \quad \text{with} \quad z_j^i = z_j(x_j^i)\phi_j^i.$$

It is easy to see that

$$v - w = \sum_{j=1}^{J} z_j = \sum_{j=1}^{J} \sum_{i=1}^{n_j} z_j^i.$$

From (34) and the fact that $w \geq 0$, it is true that

$$z_j^i + w \geq 0 \quad \forall i, j \quad \text{which means that} \quad z_j^i + w \in K.$$

Using the approximation properties (27)–(28), the following estimate is correct, see [TX01, Tai00]:

$$\sum_{j=1}^{J} \sum_{i=1}^{n_j} \left| z_j^i \right|_1^2 = \sum_{j=1}^{J} \sum_{i=1}^{n_j} \left| z_j(x_j^i) \right|^2 \left| \phi_j^i \right|_1^2 \leq C \sum_{j=1}^{J} h_j^{d-2} \sum_{i=1}^{n_j} \left| z_j(x_j^i) \right|^2$$
$$\leq C \sum_{j=1}^{J} h_j^{-2} \left| z_j \right|_0^2 \leq \tilde{c}_d \sum_{j=1}^{J} h_j^{-2} h_{j-1}^2 \left| v - w \right|_1^2 \leq \tilde{c}_d \gamma^{-2} J |v - w|_1^2.$$

The estimation for C_2 is the same as for the unconstrained case [Tai00]. Thus for the multigrid decomposition (33) we have

$$C_1 = \tilde{c}_d \gamma^{-1} J^{\frac{1}{2}} = \tilde{c}_d \gamma^{-1} |\log h|^{\frac{1}{2}}, \quad C_2 = C(1 - \gamma^d)^{-1}. \tag{35}$$

In the above, γ is the mesh ratio for the multigrid method and d is the dimension for $\Omega \subset R^d$. Thus the assumptions (6)–(7) are valid for the multigrid decomposition. Using Theorem 1, we see that the convergence rate for Algorithm 1 is:

$$\frac{F(u^{n+1}) - F(u)}{F(u^n) - F(u)} \leq 1 - \frac{\alpha}{1 + \tilde{c}_d \gamma^{-2} J}.$$

Some numerical tests

Numerical tests shall be done both for Algorithm 1 and Algorithm 2. However, we shall only explain some of the implementation details for Algorithm 1. The implementation for Algorithm 2 follows the similar techniques.

Define $u^{n+\frac{i}{m}} = u^n + e_i^{n+1}$ for Algorithm 1. When decomposition (22.b) is used for the finite element method, it can be seen that the subproblems we need to solve over each of the subdomains is:

$$a). \begin{cases} -\Delta u^{n+\frac{i}{m}} \geq f & \text{in } \Omega_i^c, \\ u^{n+\frac{i}{m}} = u^n & \text{on } \partial\Omega_i^c, \\ u^{n+\frac{i}{m}} \geq \psi & \text{in } \Omega_i^c. \end{cases} \quad \text{or} \quad b). \begin{cases} -\Delta e_i^{n+1} \geq f + \Delta u^n & \text{in } \Omega_i^c, \\ e_i^{n+1} = 0 & \text{on } \partial\Omega_i^c, \\ e_i^{n+1} \geq \psi - u^n & \text{in } \Omega_i^c. \end{cases} \tag{36}$$

It is better to solve (36.a) then get $e_i^{n+1} = u^{n+\frac{i}{m}} - u^n$. If we use (36.b) to get e_i^{n+1}, then we must compute the residual $f + \Delta u^n$ over each subdomain. This does not require extra cost for the parallel algorithm 1 as the residual is needed for the coarse mesh subproblem anyway.

However, it requires extra cost for the sequential algorithm 2 and for the case when the coarse mesh is not used. If the coarse mesh is used, we need to solve

$$e_0^{n+1} = \arg \min_{\substack{v_0 \in V_0 \\ v_0 \geq \psi - u^n}} G(v_0) \quad \text{with} \quad G(v_0) = F(u^n + v_0).$$

The unknowns for the minimization problem is the coarse mesh nodal values, but the constraint $v_0 \geq \psi - u^n$ is imposed over all the fine mesh nodes. This is not an easy problem to solve. In our implementation, we have used the Augmented Lagrangian method to minimize the functional and at the same time to impose the constraint over all the fine mesh nodes.

For the multigrid decomposition (33), each subproblem (3) is one dimensional. We just need to solve

$$\tilde{e}_{i,j}^{n+1} = \arg \min_{v_j^i \in \mathcal{M}_j^i} G(v_j^i) \quad \text{with} \quad G(v_j^i) = F(u^n + v_j^i) \tag{37}$$

and then project the value above the one dimensional constraint, i.e.

$$e_{i,j}^{n+1}(x_j^i) = \max_{x \in \text{support}(\phi_j^i)} \left(\tilde{e}_{i,j}^{n+1}(x_j^i),\ \frac{\psi(x) - u^n(x)}{\phi_j^i(x)} \right). \tag{38}$$

The solving of (37) is the same as the unconstrained case. The only extra thing we need to do is the projection given in (38).

For the test results, we shall solve the obstacle problem on $\Omega = [-2, -2] \times [-2, 2]$ with $f = 0$. The obstacle is $\psi(x, y) = \sqrt{x^2 + y^2}$ when $x^2 + y^2 \leq 1$ and $\psi(x, y) = -1$ elsewhere. This problem has an analytical solution [Tai00]. Note that the obstacle function ψ is not even in $H^1(\Omega)$ due to the discontinuity. Even for such a difficult problem, uniform linear convergence has been observed in our experiments. In the implementations, the non-zero obstacle can be shifted to the right hand side.

We will try both sequential and parallel domain decomposition. In the plots, en is the error between the computed solution and the true FEM solution in the energy norm. In all the computations, u_0 is taken to be $\psi + 100$. The domain decomposition solvers all use a two-element overlap and the subproblems are solved by an augmented Lagrangian iterative method. The mesh is discretized by letting $h = 4/64$ and $H = 4/8$. So there are 64 subdomains. The convergence-results are shown in Figure 1. In the figure, we also compare the convergence with the two corresponding algorithms of [Tai00]. It seems that the algorithms here has the same convergence rate as the one of [Tai00]. However, the B-sequential algorithm, which refers to Algorithm 2, seems to be slightly faster than the C-sequential algorithm which refers to that of [Tai00].

For the multigrid method, we have only tested the sequential algorithms. We use a V-cycle method. This is equivalent to repeat the one dimensional subspace once more in the decomposition (33) and order them properly. The convergence for 5, 6 and 7 levels are shown in Figure 2. The convergence rate is about 0.6 for all the three different levels.

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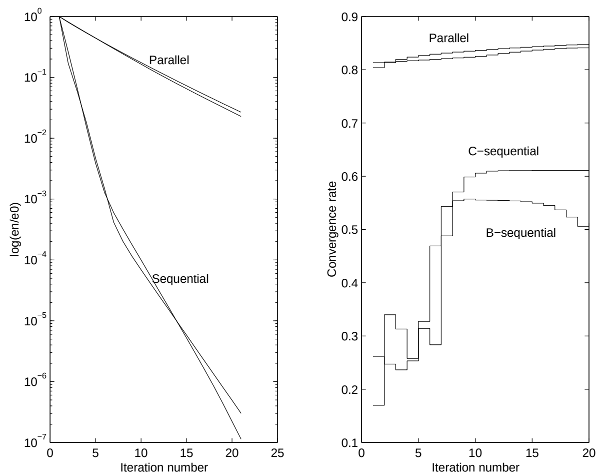

Figure 1: Domain decomposition. The B solver is Algorithm 2 and the C solver is the corresponding algorithm of [Tai00].

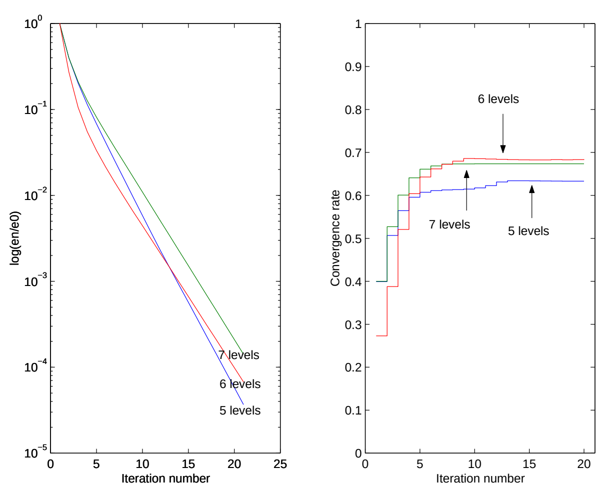

Figure 2: Convergence rate of the multigrid solver with several different levels

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