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6 Domain decomposition and fictitious domain methods with distributed Lagrange multipliers

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Introduction

In this paper we consider three applications of the distributed Lagrange multiplier technique [DGH+92, GHJ+97, GK98] to design new domain decomposition and fictitious domain methods for the diffusion equation

$$-\nabla(a\nabla u) = f, \qquad x \in \Omega, \tag{1}$$

in a bounded 2D/3D polygonal domain with the homogeneous Dirichlet boundary condition

$$u = 0, \qquad x \in \partial\Omega, \tag{2}$$

and a piece-wise constant diffusion coefficient a.

The above restrictions are imposed for the sake of simplicity. The generalizations of the algorithms and theoretical results to more complicated equations, domains, and boundary conditions are obvious.

Let Ω_h be a triangular/tetrahedral partitioning of Ω, and V_h be the corresponding piece-wise linear finite element subspace of $H_0^1(\Omega)$. We shall always assume in this paper that Ω_h is a shape-regular mesh. Then the classical finite element method

$$u^h \in V_h\colon \quad a(u_h,\, v) = l(v) \qquad \forall v \in V_h \tag{3}$$

where

$$a(u,\, v) = \int\limits_\Omega a\nabla u \cdot \nabla v \,\mathrm{d}x \quad \text{and} \quad l(v) = \int\limits_\Omega f v \,\mathrm{d}x,$$

results in the system of linear algebraic equations

$$A\bar{u} = \bar{f} \tag{4}$$

with a symmetric positive definite matrix $A \in \Re^{n\times n}$, $n = \dim V_h$, and a vector $\bar{f} \in \Re^n$. We also denote by M the mass matrix and by $\hat{M}$ the lumped mass matrix, i.e. $\hat{M}$ is diagonal and $M\bar{e} = \hat{M}\bar{e}$, $\bar{e}^T = (1, \dots, 1)$, $\bar{e} \in \Re^n$.

For $\Omega_{1,h}$ and $\Omega_{2,h}$ being non-overlapping subdomains of Ω_h such that $\Omega_h = \Omega_{1,h} \cup \Omega_{2,h}$, we denote by A_1 and A_2 the corresponding stiffness matrices and by $M_1(\hat{M}_1)$ and $M_2(\hat{M}_2)$ the corresponding mass (lumped mass) matrices. The matrices A, M and $\hat{M}$ can be introduced by subassembling of matrices A_i, M_i, $\hat{M}_i$ with the same subassembling matrices N_i, $i = 1, 2$, respectively. For instance,

$$\begin{aligned} A &= N_1\, A_1\, N_1^T + N_2\, A_2\, N_2^T, \\ \hat{M} &= N_1\, \hat{M}_1\, N_1^T + N_2\, \hat{M}_2\, N_2^T. \end{aligned}$$

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Domain decomposition for composite materials

Let Ω be a rectangle and ω_i, $i = \overline{1,m}$, $m \geq 1$, be open non-overlapping polygonal subdomains of Ω, i.e. $\omega_i \cup \omega_j = \emptyset$ for $i \neq j$ and $\partial\omega_i \cap \partial\Omega = \emptyset$, $i, j = \overline{1,m}$. An example of Ω is given in Figure 1. We assume that ω_i are shape-regular, $c_1 d \leq \mathrm{diameter}(\omega_i) \leq c_2 d$ and $\mathrm{distance}(\omega_i,\, \partial\Omega) \geq c_3 d$ with some positive constants c_1, c_2, and c_3 where $d > 0$ is given. We also assume that $a = 1 + \frac{1}{\delta_i}$, $\delta_i \equiv const \in (0,\, 1]$ in ω_i, $i = \overline{1,m}$, and $a \equiv 1$ in the rest of Ω. We shall call this model example a “composite material”.

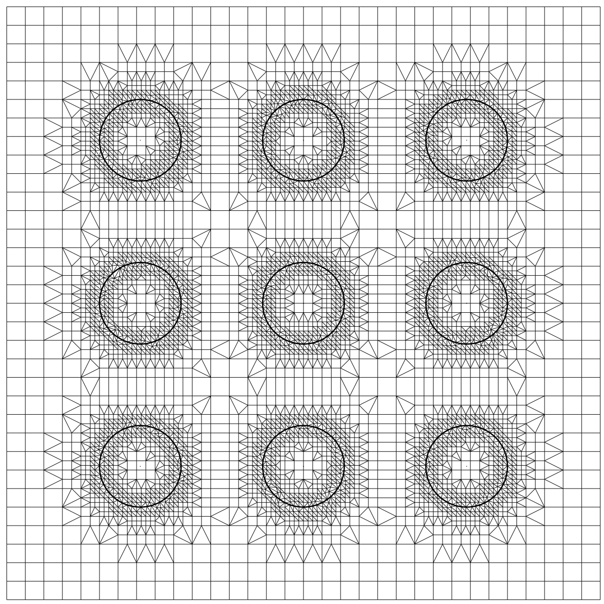

Figure 1: The computational grid.

The stiffness matrix A of system (4) can be presented in the form

$$A = A_0 + \sum_{i=1}^{m} \frac{1}{\delta_i} B_i \tag{5}$$

where

$$(B_i \bar{v},\, \bar{w}) = \int_{\omega_i} \nabla v_h \cdot \nabla w_h \,\mathrm{d}x \qquad \forall v_h, w_h \in V_h,$$

and

$$(A_0 \bar{v},\, \bar{w}) = \int_{\Omega} \nabla v_h \cdot \nabla w_h \,\mathrm{d}x \qquad \forall v_h, w_h \in V_h.$$

It is obvious that with an appropriate permutation matrix P_i we have

$$P_i B_i P_i^T = \begin{pmatrix} A_i & 0 \\ 0 & 0 \end{pmatrix}$$

where $-A_i$ is the stiffness matrix of the Laplacian for the subdomain ω_i, $1 \le i \le m$.

In [Kuz00] was proposed to replace system (4) with A in (5) by a saddle point system

$$\mathcal{A}\begin{pmatrix}\bar{u}\\ \bar{\lambda}\end{pmatrix} = \begin{pmatrix}A_0 & B^T\\ B & -C\end{pmatrix}\begin{pmatrix}\bar{u}\\ \bar{\lambda}\end{pmatrix} = \begin{pmatrix}\bar{f}\\ 0\end{pmatrix} \tag{6}$$

with

$$B^T = (B_1\ B_2\ \dots\ B_m) \in \Re^{n\times(mn)}$$

and the block diagonal matrix

$$C = \begin{pmatrix}\delta_1 B_1 & & \\ & \ddots & \\ & & \delta_m B_m\end{pmatrix} \in \Re^{(mn)\times(mn)}.$$

System (6) is equivalent to system (4) in the sense that the solution vector $\bar{u}$ to (4) coincides with the solution subvector $\bar{u}$ to (6) and vice versa. Moreover,

$$\bar{\lambda}_i - \frac{1}{\delta_i}\bar{u} \in \ker B_i$$

for any solution subvector $\bar{\lambda}_i$ to (6), $i = \overline{1,m}$.

Let a matrix $H_A = H_A^T > 0$ be spectrally equivalent to A_0^{-1}, i.e

$$c_4(H_A\bar{v},\,\bar{v}) \le (A_0^{-1}\bar{v},\,\bar{v}) \le c_5(H_A\bar{v},\,\bar{v}) \qquad \forall \bar{v} \in \Re^n$$

with positive constants c_4 and c_5 independent of the mesh Ω_h. Then the matrix

$$\mathcal{H} = \begin{pmatrix}H_A & 0\\ 0 & H_\lambda\end{pmatrix} \tag{7}$$

with

$$H_\lambda = \mathrm{diag}\{B_1^+,\,B_2^+,\,\dots,\,B_m^+\},$$

where B_i^+ denotes the generalized inverse to B_i, $i = \overline{1,m}$, was proposed in [Kuz00] as an effective preconditioner for the matrix $\mathcal{A}$ in (6). To justify the latter statement we have to consider the matrix $\mathcal{AH}$ in its invariant subspace $im\mathcal{A}$ supplied with the scalar product generated by the matrix

$$\mathcal{D} = \begin{pmatrix}H_A & 0\\ 0 & D_\lambda\end{pmatrix},$$

where

$$D_\lambda = \mathrm{diag}\{B_1,\,B_2,\,\dots,\,B_m\}.$$

It can be easily shown that $\mathcal{AH}$ is a symmetric operator in $im\mathcal{A}$ with respect to the $\mathcal{D}$-scalar product. Moreover, $im\mathcal{A} = im(\mathcal{AH})$. To this end, all non-zero eigenvalues of the matrix $\mathcal{AH}$ belong to the union of two segments $[d_1;\,d_2]$ and $[d_3;\,d_4]$ with end points

$$d_1 \le d_2 < 0 < d_3 \le d_4.$$

The condition number of $\mathcal{AH}$ with respect to the subspace $im\mathcal{A}$ and the $\mathcal{D}$-scalar product is defined by

$$\mathrm{Cond}_{\mathcal{D}}(\mathcal{AH}) = \frac{\max\{d_4;\ |d_1|\}}{\min\{d_3;\ |d_2|\}}.$$

Under all the above assumptions the following result was proved in [Kuz00].

Proposition 1

$$\mathrm{Cond}_{\mathcal{D}}(\mathcal{AH}) \leq c_6, \tag{8}$$

where c_6 is a positive constant independent of the values $\delta_1,\ \delta_2,\ \dots,\ \delta_m$ and the mesh Ω_h.

Remark 1 *In general, the constant c_6 depends on the constants c_i, $i = \overline{1,5}$.*

The implementation procedure of the preconditioner $\mathcal{H}$ is based on a simple observation that

$$B_i B_i^+ = \begin{pmatrix} Q_i & 0 \\ 0 & 0 \end{pmatrix} \tag{9}$$

where

$$Q_i \equiv A_i A_i^+.$$

The results of numerical experiments for the geometry given in Fig. 1 are presented in Table 1. For numerical experiments H_A was chosen to be the BPX-preconditioner [BPX90].

Table 1. The number of PCG iterations.

δ	13×13	34×34	76×76	160×160
1	15	16	18	18
10^{-1}	17	22	25	27
10^{-2}	19	23	27	29
10^{-3}	19	23	27	29
10^{-4}	19	23	27	29

The vectors λ_i, $i = \overline{1,m}$, in (6) can be called the discrete distributed Lagrange multipliers. They have a very simple connection with the continuous/differential distributed Lagrange multiplier. System (6) can be obtained by the straightforward finite element discretization of the variational problem: find $u \in H_0^1(\Omega)$, $\lambda_i \in H^1(\omega_i)$, $i = \overline{1,m}$, such that

$$\begin{aligned} &\int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x + \sum_{i=1}^m \int_{\omega_i} \nabla \lambda_i \cdot \nabla v \,\mathrm{d}x = \int_\Omega f v \,\mathrm{d}x, \\ &\int_{\omega_i} \nabla u \cdot \nabla \mu_i \,\mathrm{d}x - \delta_i \int_{\omega_i} \nabla \lambda_i \cdot \nabla \mu_i \,\mathrm{d}x = 0, \qquad i = \overline{1,m}, \end{aligned} \tag{10}$$

$\forall v \in H_0^1(\Omega)$, $\mu_i \in H^1(\omega_i)$, $i = \overline{1,m}$.

Fictitious domain method

The name "fictitious domain method" was originally suggested by V.K. Saul'ev in [Sau63]. The Saul'ev's idea is to replace differential problem (1)–(2) by the problem

$$\begin{aligned} -\nabla(a_\delta \nabla u_\delta) &= f_\delta, \qquad x \in \Pi, \\ u_\delta &= 0, \qquad x \in \partial\Pi, \end{aligned} \tag{11}$$

where Π is a rectangle containing the original simply-connected domain Ω,

$$a_\delta = \begin{cases} a, & x \in \Omega, \\ 1 + \frac{1}{\delta}, & x \in \Pi \setminus \bar{\Omega}, \end{cases} \qquad f_\delta = \begin{cases} f, & x \in \Omega, \\ 0, & x \in \Pi \setminus \bar{\Omega}. \end{cases}$$

It was proved that $\|u_\delta - \hat{u}\|_{H_0^1(\Omega)} \to 0$ as $\delta \to 0$ where

$$\hat{u} = \begin{cases} u, & x \in \Omega, \\ 0, & x \in \Pi \setminus \bar{\Omega}. \end{cases}$$

The form of the equation in (1) reminds us the situation considered in the previous section. If we introduce the distributed Lagrange multiplier by

$$\lambda = \frac{1}{\delta} u \tag{12}$$

in $\omega = \Pi \setminus \bar{\Omega}$, then the weak saddle point formulation reads as follows: find $u \in H_0^1(\Pi)$, $\lambda \in H^1(\omega)$, $\lambda = 0$ on $\partial\omega \cap \partial\Pi$, such that

$$\begin{aligned} &\int_\Pi \nabla u \cdot \nabla v \, \mathrm{d}x + \int_\omega \nabla\lambda \cdot \nabla v \, \mathrm{d}x = \int_\Pi f_\delta v \, \mathrm{d}x, \\ &\int_\omega \nabla u \cdot \nabla \mu \, \mathrm{d}x - \delta \int_\omega \nabla\lambda \cdot \nabla\mu \, \mathrm{d}x = 0, \end{aligned} \tag{13}$$

$\forall v \in H_0^1(\Pi)$, $\mu \in H^1(\omega)$, $\mu = 0$ on $\partial\omega \cap \partial\Pi$.

The interesting observation is that with $\delta = 0$ formulation (13) coincides with the distributed Lagrange multiplier fictitious domain method invented by R. Glowinski (see [DGH$^+$92, GHJ$^+$97]). Thus, the Glowinski's method is the closure with respect to the parameter δ of the Saul'ev's method.

The finite element discretization to (13) results in the algebraic system

$$\mathcal{A} \begin{pmatrix} \bar{u}_1 \\ \bar{u}_2 \\ \bar{\lambda} \end{pmatrix} \equiv \begin{pmatrix} A_{11} & A_{12} & 0 \\ A_{21} & A_{22} & B_{22} \\ 0 & B_{22} & -\delta B_{22} \end{pmatrix} \begin{pmatrix} \bar{u}_1 \\ \bar{u}_2 \\ \bar{\lambda} \end{pmatrix} = \begin{pmatrix} \bar{f}_1 \\ \bar{f}_2 \\ 0 \end{pmatrix} \tag{14}$$

where B_{22} stays for the stiffness matrix in subdomain ω, and

$$A_0 = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

stays for the stiffness matrix in the rectangle Π. If we present $\mathcal{A}$ in a different block form:

$$\mathcal{A} = \begin{pmatrix} A_0 & B^T \\ B & -\delta C \end{pmatrix}, \qquad C = B_{22},$$

and assume that a matrix H_A is spectrally equivalent to A_0^{-1}, then the preconditioner for $\mathcal{A}$ can be proposed in the form of the block diagonal matrix

$$\mathcal{H} = \begin{pmatrix} H_A & 0 \\ 0 & H_\lambda \end{pmatrix} \tag{15}$$

where $H_\lambda = B_{22}^{-1}$.

Assume that the norm preserving finite element extension theorem for the subdomain ω with respect to the rectangle Π holds. Then,

$$\mathrm{Cond}_{\mathcal{H}}(\mathcal{A}\mathcal{H}) \le c_7$$

where c_7 is a positive constant independent of the mesh Π_h and value of $\delta \in [0;\,1]$. In the case $\delta = 0$ the result was proved in [GK98]. For the case $\delta > 0$ one has to use technique from [Kuz00].

Overlapping domain decomposition

Let Ω_h be partitioned into two subdomains $\Omega_{1,h}$ and $\Omega_{2,h}$ such that $G_h = \Omega_{1,h} \cap \Omega_{2,h}$ is nonempty. We assume that $\mathrm{meas}(\partial G_h \cap \partial\Omega) \ge const > 0$, and the norm preserving finite element extension results from G_h into $\Omega_{1,h}$ and $\Omega_{2,h}$ hold [Wid87]. Later we shall give the algebraic interpretation of this assumption.

Let the bilinear form $a(u,\,v)$ be split into two bilinear forms [Kuz97]:

$$a(u,\,v) = a_1(u,\,v) + a_2(u,\,v) \tag{16}$$

and the linear form $l(v)$ be also splitted into two linear forms:

$$l(v) = l_1(v) + l_2(v) \tag{17}$$

where

$$a_i(u,\,v) = \int\limits_{\Omega_i} a_i \nabla u \cdot \nabla v \,\mathrm{d}x$$

with

$$a_i = \begin{cases} a, & x \in \Omega_i \setminus G, \\ a/2, & x \in G, \end{cases}$$

and

$$l_i(v) = \int\limits_{\Omega_i} \alpha_i f v \,\mathrm{d}x$$

with

$$\alpha_i = \begin{cases} 1, & x \in \Omega_i \setminus G, \\ 1/2, & x \in G, \end{cases}$$

$i = 1, 2$. Then, let us define two new bilinear and linear forms by

$$\begin{aligned} \hat{a}(\bar{u}, \bar{v}) &= a_1(u_1, v_1) + a_2(u_2, v_2), \\ b(\lambda, \bar{v}) &= \int_G \nabla\lambda \cdot \nabla(v_1 - v_2)\, \mathrm{d}x, \\ \hat{l}(\bar{v}) &= l_1(v_1) + l_2(v_2) \end{aligned} \tag{18}$$

where

$$\bar{v} = \begin{pmatrix} v_1, \\ v_2 \end{pmatrix}, \quad v_i \in V_i = \left\{ v\colon\ v \in H^1(\Omega_i),\ v = 0 \ \text{ on } \ \partial\Omega \cap \partial\Omega_i \right\}, \quad i = 1, 2,$$

and

$$\lambda \in V_\lambda = \left\{ \lambda\colon\ \lambda \in H^1(G),\ \lambda = 0 \ \text{ on } \ \partial\Omega \cap \partial G \right\}.$$

Then, the weak formulation of (1) based on the above overlapping decomposition with distributed Lagrange multipliers can be given by: find $\bar{u} \in \hat{V} = V_1 \times V_2$, $\lambda \in V_\lambda$ such that

$$\begin{aligned} \hat{a}(\bar{u}, \bar{v}) + b(\lambda, \bar{v}) &= \hat{l}(v), \\ b(\bar{u}, \mu) &= 0 \end{aligned} \tag{19}$$

$\forall \bar{v} \in \hat{V}$, $\mu \in V_\lambda$.

The finite element discretization of (19) can be suggested with the same formulae by replacing $\hat{V}$ and V_λ by $\hat{V}_h$ and $V_{\lambda,h}$ which are the traces of the finite element space V_h onto $\Omega_{1,h}$, $\Omega_{2,h}$ and G_h, respectively. The finite element discretization of (19) results in the system of algebraic equations

$$\mathcal{A} \begin{pmatrix} \bar{u} \\ \bar{\lambda} \end{pmatrix} \equiv \begin{pmatrix} A_1 & 0 & B_1^T \\ 0 & A_2 & B_2^T \\ B_1 & B_2 & 0 \end{pmatrix} \begin{pmatrix} \bar{u}_1 \\ \bar{u}_2 \\ \bar{\lambda} \end{pmatrix} = \begin{pmatrix} \bar{f}_1 \\ \bar{f}_2 \\ 0 \end{pmatrix}, \tag{20}$$

where

$$A_1 = \begin{pmatrix} A_{11} & A_{1G} \\ A_{G1} & A_{GG}^{(1)} \end{pmatrix} \qquad A_2 = \begin{pmatrix} A_{GG}^{(2)} & A_{G2} \\ A_{2G} & A_{22} \end{pmatrix},$$

$$B_1^T = \begin{pmatrix} 0 \\ B_G \end{pmatrix}, \qquad B_2^T = \begin{pmatrix} B_G \\ 0 \end{pmatrix}.$$

Here B_G is defined by

$$(B_G \bar{\lambda}, \bar{\mu}) = \int_G \nabla\lambda_h \cdot \nabla\mu_h\, \mathrm{d}x, \qquad \forall \lambda_h, \mu_h \in V_{\lambda,h}, \tag{21}$$

i.e. $-B_G$ is the stiffness matrix for the Laplacian in the subdomain G_h.

We introduce a preconditioner $\mathcal{H}$ for $\mathcal{A}$ in the form of a block diagonal matrix:

$$\mathcal{H} = \begin{pmatrix} H_1 & 0 & 0 \\ 0 & H_2 & 0 \\ 0 & 0 & H_\lambda \end{pmatrix}, \tag{22}$$

where H_i is spectrally equivalent to A_i^{-1}, $i = 1, 2$, and H_λ^{-1} is spectrally equivalent to the Schur complement matrix

$$S_\lambda = B_1\, A_1^{-1}\, B_1^T + B_2\, A_2^{-1}\, B_2^T\,. \tag{23}$$

We have plenty of choices for H_1 and H_2, for instance, multigrid preconditioner. The question is only about a choice for H_λ.

The assumption about the norm preserving finite element extension results (in the context of the above method) is equivalent to the assumption that the matrix B_G is spectrally equivalent to matrices

$$S_G^{(i)} = A_G^{(i)} - A_{Gi}\, A_{ii}^{-1}\, A_{iG}, \qquad i = 1, 2.$$

In this case simple transformations show that the matrix S_λ is spectrally equivalent to the matrix B_G. The conclusion is obvious: we have to choose

$$H_\lambda = B_G^{-1}.$$

Implementation procedure for H_λ is very simple due to the formulae

$$\mathcal{H}\mathcal{A} = \begin{pmatrix} H_1 & 0 & 0 \\ 0 & H_2 & 0 \\ 0 & 0 & I_\lambda \end{pmatrix} \begin{pmatrix} A_1 & 0 & B_1^T \\ 0 & A_2 & B_2^T \\ \tilde{B}_1 & \tilde{B}_2 & 0 \end{pmatrix},$$

where

$$\tilde{B}_1 = (0\ I_\lambda) \quad \text{and} \quad \tilde{B}_2 = (I_\lambda\ 0)\,.$$

Proposition 2 *Under the assumptions made, the eigenvalues of the matrix $\mathcal{H}\mathcal{A}$ belong to the union of two segments $[d_1;\, d_2]$, $[d_3;\, d_4]$ with the end points $d_1 \le d_2 < 0 < d_3 \le d_4$ independent of the mesh Ω_h.*

Remark 2 *The values of d_1, d_2, d_3 and d_4 from Proposition 2 depend on the constants of spectral equivalence H_i and A_i, as well as B_G and $S_G^{(i)}$, $i = 1, 2$.*

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