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15 Mortar spectral element discretization of Darcy's equations

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Introduction

Darcy's equations model the filtration of an incompressible viscous fluid in porous media. However, exactly the same equations are involved in the mixed formulation of the Laplace equation with Neumann boundary conditions and also in the projection–diffusion algorithm of Chorin [Cho68] and Temam [Tem68] for solving the time-dependent Navier–Stokes equations. So proposing discretizations of this problem which are both accurate and efficient, seems rather important. We first write its variational formulation, which involves the domain of the divergence operator, and prove that it is well-posed. We describe a spectral discretization of the problem that relies on the mortar domain decomposition technique introduced by Bernardi, Maday and Patera [BMP94], since it combines the accuracy of standard spectral methods with the advantage of handling complex geometries via the mortar algorithm. We prove the convergence of the discrete solution towards the exact one and derive error estimates.

Detailed proofs of the results presented in this paper can be found in [ABB03], and numerical experiments are under consideration.

Darcy's equations and their variational formulation

Let Ω be a bounded connected domain in $\mathbb{R}^d$, $d = 2$ or 3, with a Lipschitz–continuous boundary, and let $\mathbf{n}$ denote the unit normal vector outward to Ω. Darcy's equations in this domain write

$$\begin{aligned} \mathbf{u} + \mathbf{grad}\, p = \mathbf{f} \qquad & \text{in } \Omega, \\ \operatorname{div} \mathbf{u} = 0 \qquad & \text{in } \Omega, \\ \mathbf{u} \cdot \mathbf{n} = 0 \qquad & \text{on } \partial\Omega, \end{aligned} \tag{1}$$

where the unknowns are the velocity $\mathbf{u}$ and the pressure p. In order to write the variational formulation of problem (1), we first consider the space

$$H(\operatorname{div}, \Omega) = \left\{ \mathbf{v} \in L^2(\Omega)^d;\ \operatorname{div} \mathbf{v} \in L^2(\Omega) \right\}, \tag{2}$$

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provided with the natural norm

$$\|\mathbf{v}\|_{H(\mathrm{div},\Omega)} = \big(\|\mathbf{v}\|^2_{L^2(\Omega)^d} + \|\mathrm{div}\,\mathbf{v}\|^2_{L^2(\Omega)}\big)^{\frac{1}{2}}. \tag{3}$$

We note that $H(\mathrm{div},\Omega)$ is a Hilbert space and we recall from [GR86](Chap. I, Thm 2.4) that the space $\mathcal{D}(\overline{\Omega})^d$ of restrictions of infinitely differentiable functions on $I\!R^d$ to $\overline{\Omega}$ is dense in $H(\mathrm{div},\Omega)$. As a consequence, the trace operator: $\mathbf{v} \mapsto \mathbf{v} \cdot \mathbf{n}$, defined from the formula

$$\forall \varphi \in H^1(\Omega), \quad \langle \mathbf{v} \cdot \mathbf{n}, \varphi \rangle = \int_\Omega \big(\mathbf{v} \cdot \mathbf{grad}\,\varphi + (\mathrm{div}\,\mathbf{v})\varphi\big)(\mathbf{x})\,d\mathbf{x} \tag{4}$$

is continuous from $H(\mathrm{div},\Omega)$ onto the dual space $H^{-\frac{1}{2}}(\partial\Omega)$ of $H^{\frac{1}{2}}(\partial\Omega)$. So, we can now define the subspace

$$H_0(\mathrm{div},\Omega) = \big\{\mathbf{v} \in H(\mathrm{div},\Omega);\ \mathbf{v} \cdot \mathbf{n} = 0 \text{ on } \partial\Omega\big\}, \tag{5}$$

which is also a Hilbert space and is the closure for the norm defined in (3) of the space $\mathcal{D}(\Omega)^d$ of functions in $\mathcal{D}(\overline{\Omega})^d$ with a compact support in Ω. Finally, we introduce the space

$$L^2_0(\Omega) = \big\{q \in L^2(\Omega);\ \int_\Omega q(\mathbf{x})\,d\mathbf{x} = 0\big\}. \tag{6}$$

The variational formulation of problem (1) now reads
Find $(\mathbf{u}, p)$ *in* $H_0(\mathrm{div},\Omega) \times L^2_0(\Omega)$ *such that*

$$\begin{aligned} &\forall \mathbf{v} \in H_0(\mathrm{div},\Omega), \quad a(\mathbf{u},\mathbf{v}) + b(\mathbf{v},p) = \int_\Omega \mathbf{f}(\mathbf{x}) \cdot \mathbf{v}(\mathbf{x})\,d\mathbf{x}, \\ &\forall q \in L^2_0(\Omega), \quad b(\mathbf{u},q) = 0, \end{aligned} \tag{7}$$

where the bilinear forms $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ are defined by

$$a(\mathbf{u},\mathbf{v}) = \int_\Omega \mathbf{u}(\mathbf{x}) \cdot \mathbf{v}(\mathbf{x})\,d\mathbf{x}, \qquad b(\mathbf{v},q) = -\int_\Omega (\mathrm{div}\,\mathbf{v})(\mathbf{x})q(\mathbf{x})\,d\mathbf{x}. \tag{8}$$

From the density of $\mathcal{D}(\Omega)^d$ in $H_0(\mathrm{div},\Omega)$, it is readily checked that problem (7) is equivalent to problem (1). Problem (7) is of saddle-point type, and the arguments for proving its well-posedness are given in [GR86] (Chap. I, Thm 4.1). First, the bilinear forms $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ are continuous on $H_0(\mathrm{div},\Omega) \times H_0(\mathrm{div},\Omega)$ and $H_0(\mathrm{div},\Omega) \times L^2_0(\Omega)$, respectively. Second, let V stand for the kernel

$$V = \big\{\mathbf{v} \in H_0(\mathrm{div},\Omega);\ \forall q \in L^2_0(\Omega),\ b(\mathbf{v},q) = 0\big\}, \tag{9}$$

or, equivalently,

$$V = \big\{\mathbf{v} \in H_0(\mathrm{div},\Omega);\ \mathrm{div}\,\mathbf{v} = 0 \text{ in } \Omega\big\}. \tag{10}$$

The following ellipticity property is then obvious

$$\forall \mathbf{v} \in V, \quad a(\mathbf{v},\mathbf{v}) = \|\mathbf{v}\|^2_{H(\mathrm{div},\Omega)}. \tag{11}$$

Third, the following inf-sup condition, for a constant $\beta > 0$,

$$\forall q \in L_0^2(\Omega), \qquad \sup_{\mathbf{v} \in H_0(\mathrm{div},\Omega)} \frac{b(\mathbf{v}, q)}{\|\mathbf{v}\|_{H(\mathrm{div},\Omega)}} \geq \beta \, \|q\|_{L^2(\Omega)}, \tag{12}$$

is derived by taking $\mathbf{v}$ equal to $\mathbf{grad}\,\varphi$, where φ is the solution of the Laplace equation with data q and homogeneous Neumann boundary conditions. Combining all this leads to the following statement.

Proposition 1. *For any data* $\mathbf{f}$ *in* $L^2(\Omega)^d$*, problem* (7) *has a unique solution* $(\mathbf{u}, p)$ *in* $H_0(\mathrm{div}, \Omega) \times L_0^2(\Omega)$.

Unfortunately, even for smooth data, the solution of problem (7) is not very regular. For any data $\mathbf{f}$ in $L^2(\Omega)^d$ such that $\mathbf{curl}\,\mathbf{f}$ belongs to $L^2(\Omega)^{2d-3}$, the solution $(\mathbf{u}, p)$ belongs to $H^s(\Omega)^d$ for $s = \frac{1}{2}$ in the general case, $s = 1$ if Ω is convex and some $s > \frac{1}{2}$ if Ω is a polygon or polyhedron (we refer to [Cos90], [Dau92] and [ABDG98] for these results).

Remark: Another variational formulation of problem (1) exists, where the spaces $H_0(\mathrm{div}, \Omega)$ and $L_0^2(\Omega)$ are replaced by $L^2(\Omega)^d$ and $H^1(\Omega) \cap L_0^2(\Omega)$, respectively. Then, the boundary conditions in (1) are enforced in a variational way. However this second formulation does not seem appropriate when Darcy's system appears in the discretization of the Stokes problem, since the pressure in this problem does not belong to $H^1(\Omega)$ in most cases when Ω is a non convex polygon or polyhedron.

The mortar spectral element discrete problem

From now on, in view of applying the mortar element method to our problem, we assume that Ω admits a disjoint decomposition into a finite number of (open) rectangles in dimension $d = 2$, rectangular parallelepipeds in dimension $d = 3$:

$$\overline{\Omega} = \bigcup_{k=1}^{K} \overline{\Omega}_k \qquad \text{and} \qquad \Omega_k \cap \Omega_{k'} = \emptyset, \quad 1 \leq k \neq k' \leq K. \tag{13}$$

We make the further assumption that the intersection of each $\partial\Omega_k$ with $\partial\Omega$, if not empty, is a corner, a whole edge or a whole face of Ω_k. We denote by $\mathbf{n}_k$, $1 \leq k \leq K$, the unit normal vectors outward to Ω_k. We introduce the skeleton $\mathcal{S}$ of the decomposition, $\mathcal{S} = \cup_{k=1}^{K} \partial\Omega_k \setminus \partial\Omega$. According to the ideas in [BMP94], we choose a disjoint decomposition of this skeleton into mortars:

$$\overline{\mathcal{S}} = \bigcup_{m=1}^{M} \overline{\gamma}_m \qquad \text{and} \qquad \gamma_m \cap \gamma_{m'} = \emptyset, \quad 1 \leq m \neq m' \leq M, \tag{14}$$

where each γ_m is a whole edge in dimension $d = 2$, face in dimension $d = 3$, of a subdomain Ω_k, denoted by $\Omega_{k(m)}$. To describe the discrete spaces, for each nonnegative integer n, we define on each Ω_k, resp. on each edge or face Γ of Ω_k, the space $\mathbb{P}_n(\Omega_k)$, resp. $\mathbb{P}_n(\Gamma)$, of restrictions to Ω_k, resp. Γ, of polynomials with d, resp. $d-1$, variables and degree $\leq n$ with respect to each variable. The discretization parameter δ is then a K–tuple $(N_1, \dots, N_K)$ of integers $N_k \geq 2$. We first introduce the space $M_\delta(\Omega)$ of discrete pressures:

$$M_\delta(\Omega) = \left\{ q_\delta \in L^2_0(\Omega);\, q_{\delta\,|\Omega_k} \in \mathbb{P}_{N_k-2}(\Omega_k),\, 1 \leq k \leq K \right\}. \tag{15}$$

Next, in analogy with the standard definition of the mortar approximation of $H^1(\Omega)$ [BMP94], we define the discrete space $X_\delta(\Omega)$ which approximates $H_0(\mathrm{div}, \Omega)$. It is the space of functions $\mathbf{v}_\delta$ such that:

- their restrictions $\mathbf{v}_{\delta\,|\Omega_k}$ to each Ω_k, $1 \leq k \leq K$, belong to $\mathbb{P}_{N_k}(\Omega_k)^d$,
- their normal traces $\mathbf{v}_\delta \cdot \mathbf{n}$ vanish on $\partial\Omega$,
- the mortar function φ being defined on each γ_m, $1 \leq m \leq M$, by

$$\varphi_{|\gamma_m} = \mathbf{v}_{\delta\,|\Omega_{k(m)}} \cdot \mathbf{n}_{k(m)}, \tag{16}$$

the following matching condition holds on each edge or face Γ of Ω_k, $1 \leq k \leq K$, which is not a mortar:

$$\forall \chi \in \mathbb{P}_{N_k-2}(\Gamma), \quad \int_\Gamma (\mathbf{v}_{\delta\,|\Omega_k} \cdot \mathbf{n}_k + \varphi)(\tau)\, \chi(\tau)\, d\tau = 0. \tag{17}$$

Remark: The space X_δ is not contained in $H(\mathrm{div}, \Omega)$ since the matching conditions on the normal derivative through the interfaces are only enforced in a weak way. So the discretization is nonconforming. Starting from the standard Gauss–Lobatto formula on $]-1, 1[$, we define on each Ω_k and in each direction:

- the nodes x_i^k and y_i^k, and the weights $\rho_i^{x,k}$ and $\rho_i^{y,k}$, $0 \leq i \leq N_k$, in the case of dimension $d = 2$,
- the nodes x_i^k, y_i^k and z_i^k, and the weights $\rho_i^{x,k}$, $\rho_i^{y,k}$ and $\rho_i^{z,k}$, $0 \leq i \leq N_k$, in the case of dimension $d = 3$.

A discrete product is then introduced on each Ω_k, according if $d = 2$ or 3, by

$$(u_\delta, v_\delta)^k_\delta = \begin{cases} \sum_{i=0}^{N_k} \sum_{j=0}^{N_k} u_\delta(x_i^k, y_j^k) v_\delta(x_i^k, y_j^k)\, \rho_i^{x,k} \rho_j^{y,k} \\[2ex] \sum_{i=0}^{N_k} \sum_{j=0}^{N_k} \sum_{p=0}^{N_k} u_\delta(x_i^k, y_j^k, z_p^k) v_\delta(x_i^k, y_j^k, z_p^k)\, \rho_i^{x,k} \rho_j^{y,k} \rho_p^{z,k}. \end{cases} \tag{18}$$

The global discrete product on Ω:

$$(u_\delta, v_\delta)_\delta = \sum_{k=1}^{K} (u_\delta, v_\delta)^k_\delta, \tag{19}$$

coincides with the scalar product of $L^2(\Omega)$ for all functions u_δ and v_δ such that each product $(u_\delta v_\delta)_{|\Omega_k}$, $1 \leq k \leq K$, belongs to $\mathbb{P}_{2N_k-1}(\Omega_k)$. The discrete problem is now built from the variational formulation (7). For any continuous data $\mathbf{f}$ on $\overline{\Omega}$, it reads

Find $(\mathbf{u}_\delta, p_\delta)$ *in* $X_\delta(\Omega) \times M_\delta(\Omega)$ *such that*

$$\begin{aligned} &\forall \mathbf{v}_\delta \in X_\delta(\Omega), \quad a_\delta(\mathbf{u}_\delta, \mathbf{v}_\delta) + b_\delta(\mathbf{v}_\delta, p_\delta) = (\mathbf{f}, \mathbf{v}_\delta)_\delta, \\ &\forall q_\delta \in M_\delta(\Omega), \quad b_\delta(\mathbf{u}_\delta, q_\delta) = 0, \end{aligned} \tag{20}$$

where the bilinear forms $a_\delta(\cdot, \cdot)$ and $b_\delta(\cdot, \cdot)$ are defined by

$$a_\delta(\mathbf{u}_\delta, \mathbf{v}_\delta) = (\mathbf{u}_\delta, \mathbf{v}_\delta)_\delta, \qquad b_\delta(\mathbf{v}_\delta, q_\delta) = -(\mathrm{div}\, \mathbf{v}_\delta, q_\delta)_\delta. \tag{21}$$

Note however that, thanks to the exactness property of the quadrature formula, we have

$$\forall \mathbf{v}_\delta \in X_\delta(\Omega), \forall q_\delta \in M_\delta(\Omega), \quad b_\delta(\mathbf{v}_\delta, q_\delta) = b(\mathbf{v}_\delta, q_\delta). \tag{22}$$

To check the wellposedness of problem (20), we first state the discrete analogue of the inf-sup condition in (12), its proof combines the arguments in [ABG94] and [BBCM00]. It involves the "broken" norm

$$\|\mathbf{v}\|_{H(\mathrm{div},\cup\Omega_k)} = \Big(\sum_{k=1}^{K} \|\mathbf{v}\|^2_{H(\mathrm{div},\Omega_k)}\Big)^{\frac{1}{2}}. \tag{23}$$

Lemma 2. *There exists an integer N_D and a positive constant β_D, both depending on the decomposition of Ω but independent of δ, such that, if all the N_k are $\geq N_D$, the following inf-sup condition holds*

$$\forall q_\delta \in M_\delta(\Omega), \qquad \sup_{\mathbf{v}_\delta \in X_\delta(\Omega)} \frac{b(\mathbf{v}_\delta, q_\delta)}{\|\mathbf{v}_\delta\|_{H(\mathrm{div},\cup\Omega_k)}} \geq \beta_D \, \|q_\delta\|_{L^2(\Omega)}, \tag{24}$$

Proposition 3. *For any continuous data $\mathbf{f}$ on $\overline{\Omega}$ and if all the N_k are $\geq N_D$, problem (20) has a unique solution $(\mathbf{u}_\delta, p_\delta)$ in $X_\delta(\Omega) \times M_\delta(\Omega)$.*
Proof: Problem (20) results into a square linear system, so that it has a unique solution if and only if the only solution for $\mathbf{f} = \mathbf{0}$ is $(\mathbf{0}, 0)$. So we take $\mathbf{f}$ equal to $\mathbf{0}$. Choosing $\mathbf{v}_\delta$ equal to $\mathbf{u}_\delta$ in (20) yields that $a_\delta(\mathbf{u}_\delta, \mathbf{u}_\delta)$ is zero and, since the weights of the Gauss–Lobatto formula are positive, this imples that $\mathbf{u}_\delta$ vanishes in the $(N_k+1)^d$ nodes of a tensorized grid on each Ω_k, hence is zero. Then, $b_\delta(\mathbf{v}_\delta, p_\delta)$ is equal to zero for all $\mathbf{v}_\delta$ in X_δ, hence p_δ is zero due to (24).

A priori analysis

The main difficulty for evaluating the error on the velocity comes from the fact that the form $a_\delta(\cdot,\cdot)$ is no longer uniformly elliptic with respect to the norm $\|\cdot\|_{H(\mathrm{div},\cup\Omega_k)}$ on the discrete kernel

$$V_\delta = \big\{\mathbf{v} \in X_\delta(\Omega);\ \forall q_\delta \in M_\delta(\Omega),\ b_\delta(\mathbf{v}_\delta, q_\delta) = 0\big\}, \tag{25}$$

since V_δ is not made of exactly divergence-free functions. So the usual arguments for bounding the error does not hold, and we must evaluate "by hand" the quantity $\|\mathbf{u} - \mathbf{u}_\delta\|_{L^2(\Omega)^d}$. It involves three terms:

- the approximation error, which is easy to evaluate in dimension $d = 2$ but requires some further conformity assumptions in dimension $d = 3$,
- the error issued from numerical integration,
- the consistency error, which gives rise to a term of type (here, $[\cdot]$ denotes the jump through $\mathcal{S}$ with the appropriate sign)

$$\inf_{q_\delta \in M_\delta(\Omega)} \sup_{\mathbf{w}_\delta \in X_\delta(\Omega)} \frac{\int_{\mathcal{S}} (\mathbf{w}_\delta \cdot \mathbf{n})(\tau)\, [p - q_\delta](\tau)\, d\tau}{\|\mathbf{w}_\delta\|_{H(\mathrm{div},\cup\Omega_k)}}, \tag{26}$$

and it seems that using an inverse inequality is unavoidable to bound this term, which leads to non optimal estimates. Once the error on the velocity is evaluated, the error on the pressure is derived from the inf-sup condition (24). Let μ_δ denote the maximal ratio $N_k/N_{k'}$ for all pairs of subdomains Ω_k and $\Omega_{k'}$, $1 \leq k \neq k' \leq K$, such that $\partial\Omega_k \cap \partial\Omega_{k'}$ has a positive measure in $\mathcal{S}$.

Theorem 4. *In dimension $d = 2$, assume the data $\mathbf{f}$ such that each $\mathbf{f}_{|\Omega_k}$, $1 \leq k \leq K$, belongs to $H^{\sigma_k}(\Omega_k)^2$, $\sigma_k > 1$, and the solution $(\mathbf{u}, p)$ of problem (1) such that each $(\mathbf{u}_{|\Omega_k}, p_{|\Omega_k})$, $1 \leq k \leq K$, belongs to $H^{s_k}(\Omega_k)^2 \times H^{s_k+1}(\Omega_k)$, $s_k > 0$. If all the N_k are $\geq N_D$, the following error estimate holds between this solution $(\mathbf{u}, p)$ and the solution $(\mathbf{u}_\delta, p_\delta)$ of problem (20):*

$$\begin{aligned} &\|\mathbf{u} - \mathbf{u}_\delta\|_{L^2(\Omega)^2} + \|p - p_\delta\|_{L^2(\Omega)} \\ &\qquad \leq c \sum_{k=1}^{K} \big(\mu_\delta\, N_k^{\frac{1}{2}-s_k}\, (\|\mathbf{u}\|_{H^{s_k}(\Omega_k)^2} + \|p\|_{H^{s_k+1}(\Omega_k)}) \\ &\qquad\qquad + N_k^{-\sigma_k}\, \|\mathbf{f}\|_{H^{\sigma_k}(\Omega_k)^2}\big). \end{aligned} \tag{27}$$

This estimate is not optimal, however the same lack of optimality appears in several finite element discretizations of Darcy's equations (for instance, when Crouzeix–Raviart finite elements are employed for the approximation of the velocity). Moreover, if the parameter μ_δ is bounded independently of δ, the convergence of the method can be derived form this estimate in all polygons, thanks to the regularity results stated in Section 1.

To conclude, we recall that the decomposition (13) of Ω is said to be conforming if the intersection of all $\partial\Omega_k$ and $\partial\Omega_{k'}$, $1 \leq k < k' \leq K$, if not empty, is a whole edge in dimension $d = 2$, a whole face in dimension $d = 3$, of both Ω_k and $\Omega_{k'}$. The mortar element method does not require the conformity of the decomposition. However, if the decomposition is conforming, an approximation q_δ of the pressure p can be constructed in $M_\delta(\Omega) \cap H^1(\Omega)$, which means that the quantity in (26) vanishes for this q_δ. So the error estimate is optimal in this case.

Corollary 5. *If all assumptions of Theorem 4 hold and if, moreover,*
(i) the decomposition (13) of Ω is conforming,
(ii) in dimension $d = 3$, for $1 \leq m \leq M$, $N_{k(m)}$ is $\geq N_k$, where γ_m is the intersection of this $\overline{\Omega}_k$ and $\overline{\Omega}_{k(m)}$,
the following error estimate holds between the solution $(\mathbf{u}, p)$ of problem (1) and the solution $(\mathbf{u}_\delta, p_\delta)$ of problem (20):

$$\begin{aligned} &\|\mathbf{u} - \mathbf{u}_\delta\|_{L^2(\Omega)^d} + \|p - p_\delta\|_{L^2(\Omega)} \\ &\leq c \sum_{k=1}^{K} \big(N_k^{-s_k}\, (\|\mathbf{u}\|_{H^{s_k}(\Omega_k)^d} + \|p\|_{H^{s_k+1}(\Omega_k)}) + N_k^{-\sigma_k}\, \|\mathbf{f}\|_{H^{\sigma_k}(\Omega_k)^d}\big). \end{aligned} \tag{28}$$

Conclusion

As a conclusion, the mortar spectral element discretization of problem (1) is fully optimal in the case of a conforming decomposition. It is not for a nonconforming decomposition, however estimate (27) can be improved in this case by taking into account the local properties of conformity. It can also be noted that, for smooth data, the solution of problem (1) is regular outside a neighbourhood of the corners and edges of Ω, so that enforcing the conformity of the decomposition is more important in a neighbourhood of $\partial\Omega$ than elsewhere.

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