

Exotic coarse spaces for Schwarz methods for lower order and spectral finite elements

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ABSTRACT. Fast domain decomposition algorithms for elliptic problems are typically two-level methods. The second level enables us to construct iterative methods with convergence rates that are independent of the number of subregions. The construction of a suitable coarse, global subspace is often the most interesting part of the design and analysis. We consider several coarse spaces for lower order finite element methods and we then show that one of them has a useful analog in the case of continuous, piecewise Q_p elements defined on cubic elements. For all these methods, the condition number of the relevant iteration operator grows only logarithmically or polylogarithmically with the number of degrees of freedom associated with a subregion, even if there are arbitrarily large jumps in the coefficients across subregional interfaces.

1. Introduction

Fast domain decomposition algorithms for elliptic problems are typically two-level methods. The second level enables us to construct iterative methods with convergence rates independent of the number of subregions. The construction of a suitable coarse, global subspace is the most interesting part of the design, analysis, and practice. In this paper, we will briefly survey recent results obtained jointly with Maksymilian Dryja, Luca Pavarino, and Barry Smith; see [6, 7, 10]. For related work on multilevel methods; cf. Dryja, Sarkis, and Widlund [4]. See also the papers by Dryja, Pavarino, and Sarkis in these proceedings.

We first consider several coarse spaces for lower order, h -version finite element methods. There are polylogarithmic bounds for the condition number $\kappa(T_h)$ of the

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relevant iteration operator T_h based on such a coarse space and good local spaces. For several new methods, we have

$$\kappa(T_h) \leq C(1 + \log(H/h));$$

cf. Dryja, Smith, and Widlund [7]. C is independent of the mesh parameters and the jumps in the coefficient, but depend on the aspect ratios of the subdomains.

We then consider a Galerkin method using *spectral* elements and obtain

$$\kappa(T_p) \leq C(1 + \log(p))^2.$$

The bounds are uniform in the number of subproblems. The number of degrees of freedom for each subproblem is on the order of $(H/h)^3$ and p^3 , respectively, and the bounds of the condition numbers are thus polylogarithmic functions of these quantities. Each subproblem can be handled by a processor of a parallel or distributed computing system; the size of the local problem will be limited primarily by the amount of fast memory available to the individual processors.

In this paper, we consider only *iterative substructuring methods* and only three dimensional problems. The local problems of an iterative substructuring method communicate with their neighbors only through the boundary values of the iterates; this is different from the Schwarz methods that use overlapping subregions.

Earlier important work on iterative substructuring methods for the h-version finite element methods was carried out by Bramble, Pasciak, and Schatz, [2], Dryja [5], and Smith [12, 13]. Work on p-version finite element methods is described in Mandel [8]. Important work on the p-version for two dimensions is described in Babuška, Craig, Mandel, and Pitkäranta [1]. Results on iterative substructuring for higher order elements methods for two dimensions are included in Pavarino's thesis [9], which also contains results on several domain decomposition algorithms based on overlapping subregions.

2. The elliptic problems

We consider linear, elliptic problems on a bounded Lipschitz domain Ω in R^3 , formulated as: Find u such that

$$a(u, v) = \int_{\Omega} \rho(x) \nabla u \cdot \nabla v \, dx = f_{\Omega}(v), \forall v \in V.$$

$\rho(x) > 0$ can be discontinuous, with very different values for different substructures, but $\rho(x) = \rho_i$ on the substructure Ω_i . $\bar{\Omega} = \cup \bar{\Omega}_i$, $\Omega_i \cap \Omega_j = \emptyset, i \neq j$. V is a subspace of $H^1(\Omega)$ chosen appropriately.

The finite element space is either the space $V^h \subset V$ of continuous, piecewise linear functions on the elements, into which the substructures have been partitioned, or $V^p \subset V$, the space of continuous piecewise Q_p functions.

For the *lower order case*, we can, if so desired, use quite general substructures, which determine a coarse "triangulation". We can think of substructures as pieces of a three-dimensional jigsaw puzzle. Each such piece is a union of whole elements.

For the *spectral case*, we assume that the region is a union of elements that are cubes or images of a reference cube under reasonable smooth mappings. No element should be “too distorted”.

The finite element problems are obtained by restricting u and the test functions to the space $\tilde{V} = V^h$ or V^p .

3. Excerpts from additive Schwarz analysis

In this section, we give a brief overview of a useful method of analyzing domain decomposition methods; see Dryja et al. [6, 7] for many further details.

We decompose the finite element space \tilde{V} into subspaces V_i . We need not use a direct sum. For each subspace V_i , we introduce an orthogonal projection P_i onto V_i :

$$a(P_i u, v) = a(u, v), \quad \forall v \in V_i, \quad u \in \tilde{V},$$

or use an approximate solver defined by

$$\tilde{a}_i(T_i u, v) = a(u, v), \quad \forall v \in V_i, \quad u \in \tilde{V}.$$

T_i defines the inner product $\tilde{a}_i(\cdot, \cdot)$ and vice versa.

The major effort in proving a good bound for the condition number of the relevant operator $T = \sum T_i$, goes into proving that there exist C_0 and ω , of modest size, such that there exists a decomposition

$$\begin{aligned} u &= \sum_{i=0}^N u_i, \quad u_i \in V_i \quad \forall u \in \tilde{V}, \quad \text{with} \\ \sum_{i=0}^N \tilde{a}_i(u_i, u_i) &\leq C_0^2 a(u, u), \quad \text{and} \\ a(u_i, u_i) &\leq \omega \tilde{a}_i(u_i, u_i) \quad \forall u_i \in V_i. \end{aligned}$$

There is also a well established theory for multiplicative, Gauss-Seidel-type methods that involves the same parameters; see Bramble et al. [3] and Dryja et al. [6, 7].

For many iterative substructuring methods, a local analysis is possible. We can then compare $a^{(j)}(u, u)$, the contribution of the substructure Ω_j to the strain energy, to the contributions $\tilde{a}_i^{(j)}(u_i, u_i)$ of Ω_j to the preconditioner; we derive upper and lower bounds for

$$\frac{\sum_{i=0}^N \tilde{a}_i^{(j)}(u_i, u_i)}{a^{(j)}(u, u)}.$$

This greatly simplifies the derivation of bounds independent of the jumps of the coefficients; cf. [7]. Exactly one of the local problems, the one which corresponds to the global subspace, must have the same null space as the local finite element problem to be able to obtain bounds independent of the number of subregions.

4. Choosing a good coarse space V_0

The obvious choice for V_0 is V^H and Q_1 for the lower order and spectral case, respectively. However, if all the elements of the other subspaces vanish at the vertices of the substructures, then $\kappa(T)$ must grow linearly with H/h and p^2 , respectively, for three dimensional problems.

We therefore turn to more exotic spaces. As previously pointed out, we also gain greater freedom in choosing the shape of the subregions.

There are a number of important geometric objects: *interiors*, *faces*, *edges*, and *vertices*. We can merge edges and vertices, creating a *wire basket* W_i for each substructure. The subspaces are directly related to these objects. A set of counting functions describes it all. These functions also define a good coarse space.

The counting functions ν_i are defined on Γ_h , the set of nodes on all interfaces between the substructures, by

$$\begin{aligned}\nu_i(x) &= \text{number of } \partial\Omega_{j,h} \text{ to which } x \in \partial\Omega_{i,h} \text{ belongs} \\ \nu_i(x) &= 0, \quad x \in \Gamma_h \setminus \partial\Omega_{i,h}\end{aligned}$$

Just one basis function, μ_i , is related to each Ω_i . It is given by the pseudo inverse of $\nu_i(x)$, in the Laplace case, and by a similar formula involving the ρ_i , in the general case. (Special rules are used for boundary substructures.) The $\mu_i(x)$ are discrete harmonic in each substructure and their span defines our first good coarse space, V_C^h ; see Dryja et al. [6, 7].

The coarse component of the decomposition is selected as

$$u_0 = \sum_i \bar{u}_i \mu_i.$$

\bar{u}_i is the average of $u \in V^h$ over $\partial\Omega_{i,h}$. This is an interpolation formula, which reproduces constants.

5. Two additional good coarse spaces

Another good coarse space, V_W^h , was introduced in Barry Smith's thesis; cf. [12, 13]. It can be regarded as being *wire basket based*. The elements of this coarse space are piecewise discrete harmonic functions. Their values on all of $\partial\Omega_i$ are defined by the values on the wire basket W_i ; the value on each face F^k is constant and equals the average $\bar{u}_{\partial F^k}^h$ of the nodal values on the boundary of the face. The related interpolation formula is given, on Ω_i , by

$$I_W^h u^h = \sum_{x_l \in W_i} u^h(x_l) \varphi_l + \sum_{\partial F^k \subset \partial\Omega_i} \bar{u}_{\partial F^k}^h \theta_{F^k}.$$

Here $\theta_{F^k} = 1, x \in F^k$, and $= 0, x \in \Gamma_h \setminus F^k$, and φ_l is the discrete harmonic extension of the standard nodal basis function of x_l . A simple bilinear form is used

$$\tilde{a}_0(u^h, u^h) = (1 + \log(H/h))h \sum_i \rho_i \min_{c_i} \|\underline{u} - c_i \underline{z}^{(i)}\|_{l_2(W_i)}^2,$$

where $z_j^{(i)} = 1, \forall j$. A linear system results with only one essentially global degree of freedom, c_i , per substructure. Additionally, there is a linear system with a diagonal matrix that determines the nodal values on the wire baskets.

A different coarse space, V_M^h , for which a bound that is linear in $\log(H/h)$ has been established, is obtained by assigning independent degrees of freedom to the functions θ_{F^k} . This is a larger *face based* space; cf. also Sarkis [11]. We also need to use high quality local spaces to obtain such a strong bound.

6. A good coarse space for spectral elements

We have designed a wire basket based coarse space for the spectral elements; see Pavarino and Widlund [10] and additional forthcoming papers.

The values on the wire basket W_i of a given Q_p polynomial are extended to the faces, and then to the entire cube, by using formulas based on separation of variables. There are also additional terms similar to those involving the θ_{F^k} ; one special function, $\kappa_k(x, y, z)$, is constructed for each face of the cube. We begin with the values of the function 1 on the wire basket and extend them elsewhere by the formula just suggested. We then subtract the resulting function from 1 and obtain $\kappa(x, y, z)$. We write

$$\kappa(x, y, z) = \sum_{k=1}^6 \kappa_k(x, y, z),$$

where each $\kappa_k(x, y, z) = 0$ for five of the six faces of the cube.

The weights of the κ_k terms can be chosen, as for V_W^h , as the averages over the boundaries of the faces. The resulting space, by construction, contains the constant functions. A simple auxiliary bilinear form $\tilde{a}_0(\cdot, \cdot)$ can be constructed with only one essentially global degree of freedom per substructure.

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