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Balancing Domain Decomposition for Plates

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ABSTRACT. We show that the Neumann-Neumann preconditioner with a coarse problem can be applied to the solution of a system of linear equations arising from the thin plate problem discretized by the HCT and DKT elements. The condition number is asymptotically bounded by $\log^2(H/h)$, with H the subdomain size and h the element size. The bound is independent of coefficient jumps of arbitrary size between subdomains. Numerical results are presented.

1. Introduction

This note presents an application of the Balancing Domain Decomposition (BDD) method to the solution of linear systems of equations arising from the finite element discretization of thin plate problems. The BDD method was developed from the *Neumann-Neumann preconditioner* of De Roeck and Le Tallec [**5**] by Mandel [**9**], who has modified the algorithm by adding a *coarse problem* with few unknowns per subdomain. Solving the coarse problem in each iteration coordinates the solution between the subdomains and prevents any slow-down with an increasing number of subdomains.

The coarse problem, as introduced in [**9**], is composed of the rigid body modes of the substructures. Other modes can be added to the coarse problem to remove troublesome modes from the iterative process; in effect, these modes are resolved directly in every iteration. While the possibility of adding such modes has been known, it was not clear how to do that efficiently. This paper presents the first such example: for thin plates, these modes are the subdomain solutions for point loads applied at crosspoints (i.e., at subdomain corners).

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For the reduced Hsieh-Clough-Tocher (HCT) and Discrete Kirchoff Triangle (DKT) elements, the condition number of the algorithm is proved to grow at most as fast as $\log^2(H/h)$, where H and h are the characteristic subdomain size and element size. Numerical results confirm that the fast growth of the condition number for decreasing h is indeed prevented by the additional coarse functions. Using a result of Mandel and Brezina [**10**], it is shown that the bound does not depend on the jumps of elasticity coefficients between subdomains. We omit the proofs of technical lemmas, but the principal argument is complete.

For related work on the Neumann-Neumann preconditioner, we refer to Glowinski and Wheeler [**8**]. For a somehow different formulation of the Neumann-Neumann problem with similar bounds for second order problems we refer to Dryja and Widlund [**7**]. The BDD method was also applied to mixed problems by Cowsar, Mandel, and Wheeler [**4**].

2. Finite Element Plate Model

Let $\Omega \subset \mathbb{R}^2$ be a bounded polygonal domain decomposed into nonoverlapping subdomains $\Omega_1, \ldots, \Omega_k$. There is given a conforming triangulation $\{T\}$ of Ω such that each Ω_i is the union of some triangles (elements) from $\{T\}$. The subdomains Ω_i and the elements $\{T\}$ are shape regular. The characteristic subdomain size is H and the characteristic element size is h. Throughout the paper, C and c are generic constants that do not depend on H and h but may depend on the shape regularity of the triangulation and subdomain decomposition. The union of all subdomain boundaries is $\Gamma = \cup_{i=1}^k \partial\Omega_i$, and it consists of *edges* and *crosspoints* at the junctures of the edges. We assume that each subdomain Ω_i has at least three points that are crosspoints or in the part of the boundary $\partial\Omega_s$ where the plate is simply supported, and the three points form a triangle with angles bounded below by $1/C$.

Spectral equivalence of quadratic forms is defined by

$$a(u,u) \sim b(u,u) \iff \exists C\, \forall u : \frac{1}{C}a(u,u) \leq b(u,u) \leq Ca(u,u).$$

The domain of the forms will be always clear from the context. The Sobolev seminorms in $W^{m,p}(\Sigma)$ are denoted as usual by $|u|_{m,p,\Sigma}$, $|\cdot|$ is the Euclidean norm, and $P_p(\Sigma)$ is the space of all polynomial functions of order p on Σ.

We solve the problem of finding the displacement of a thin plate occupying the domain Ω, clamped on $\Omega_c \subset \partial\Omega$ and simply supported on $\Omega_s \subset \partial\Omega$. Plate elements used in engineering practice have typically three degrees of freedom per node, corresponding to the transversal displacement u and the rotations $\vec{\theta} = (\theta_\alpha)$, $\alpha = 1, 2$. Under the Kirchoff hypothesis

$$\vec{\theta} = \nabla u, \tag{1}$$

the problem to be solved is to find the transversal displacement u so that

$$u \in \mathbb{H}(\Omega) : \quad a(u,v) = L(v), \quad \forall v \in \mathbb{H}(\Omega), \tag{2}$$

where

$$\begin{aligned} a(u,v) &= \int_\Omega \varepsilon(\vec{\theta}(u)) : K : \varepsilon(\vec{\theta}(v)), \\ L(v) &= \int_\Omega fv + \int_{\partial\Omega-\partial\Omega_c} m_g \partial_n v + \int_{\partial\Omega-\partial\Omega_s} gv, \\ \mathbb{H}(\Omega) &= \{v \in H^2(\Omega), v = 0 \text{ on } \partial\Omega_s, \quad \partial_\nu v = 0 \text{ on } \partial\Omega_c\}, \end{aligned}$$

where ε is the linearized strain tensor. The plate flexural stiffness tensor K is assumed to be symmetric, measurable, and, on each subdomain Ω_i, uniformly positive definite and bounded:

$$\varepsilon : K(x) : \varepsilon \sim \rho_i \varepsilon : \varepsilon, \quad \rho_i > 0, \qquad \forall x \in \Omega_i.$$

Then one has the spectral equivalence

$$a(u,u) \sim \sum_{i=1}^{k} \rho_i \, |u|^2_{2,2,\Omega_i}, \tag{3}$$

which will be the starting point of our investigations. The numbers ρ_i have interpretation as the *relative stiffnesses* of the subplates Ω_i.

We first consider a discretization of the problem (2) by the reduced Hsieh-Clough-Tocher (HCT) triangle. Here we only note that the HCT element is C^1 continuous, satisfies $u \in P_3(\sigma)$, $\partial_\nu u \in P_1(\sigma)$ on each side σ of the element, and is composed of three P_3 subtriangles; for more details on the HCT element, see **[3]**. Denote by I_{HCT} the interpolation operator associated with the HCT elements. The finite element discrete problem is obtained by replacing the space $\mathbb{H}$ in (2) by the finite element space $\mathbb{H}_h = \mathbb{H}_h(\Omega) = \mathbb{H} \cap \operatorname{Im} I_{\mathrm{HCT}}$. The value of $I_{\mathrm{HCT}}U$ on a side of T depends on the degrees of freedom on the side. Hence, by abuse of notation, we also note

$$I_{\mathrm{HCT}} : V_i \to \mathbb{H}_h(\partial\Omega_i), \qquad I_{\mathrm{HCT}} : V \to \mathbb{H}_h(\Gamma).$$

Here V_i and V are the spaces of vectors of degrees of freedom on $\partial\Omega_i$ and Γ, respectively. $\mathbb{H}_h(\partial\Omega_i)$ and $\mathbb{H}_h(\Gamma)$ are the spaces of traces of functions from $\mathbb{H}_h$ on $\partial\Omega_i$ and Γ, respectively.

The local stiffness matrices, defined by

$$X^t A_T Y = \int_T \varepsilon(\vec{\theta}(I_{\mathrm{HCT}}X)) : K : \varepsilon(\vec{\theta}(I_{\mathrm{HCT}}Y)),$$

satisfy the spectral equivalence property

$$U^t A_T U \sim \rho_i \|\nabla I_{\mathrm{HCT}} U\|^2_{1,T}. \tag{4}$$

The theory presented in this paper applies to the HCT element and to any element with the degrees of freedom u, θ_1, θ_2 at each vertex satisfying (4). The Discrete Kirchoff Triangle (DKT) element is an example which enforces (1) only along each side of the element T **[1]**. The proof of (4) for the DKT element as well as for stabilized Reissner-Mindlin elements will be presented elsewhere. Quadrilateral elements may be treated as two triangles for which (4) holds.

3. Formulation of the Algorithm

We recall the algorithm following [**9, 10**]. The local stiffness matrix corresponding to subdomain Ω_i is A_i and U_i is the corresponding vector of degrees of freedom. Let $\bar{N}_i$ denote the matrix with entries 0 or 1 mapping the degrees of freedom U_i into global degrees of freedom: $U_i = \bar{N}_i^t U$. Write

$$A_i = \begin{pmatrix} \bar{A}_i & B_i \\ B_i^t & \dot{A}_i \end{pmatrix}, \qquad \bar{N}_i = (\bar{N}_i, \dot{N}_i),$$

where the first block corresponds to degrees of freedom on Γ. Eliminating the remaining degrees of freedom, one obtains the reduced system

$$SX = B, \tag{5}$$

for unknown values X of the degrees of freedom on Γ, posed in the space V. The matrix S is the Schur complement, defined by

$$S = \sum_{i=1}^{k} \bar{N}_i S_i \bar{N}_i^t, \qquad S_i = A_i - B_i \dot{A}_i^{-1} B_i^t\ .$$

The reduced system (5) is solved by a preconditioned conjugate gradient algorithm. To define the preconditioner, we need auxiliary matrices D_i and Z_i such that

$$\sum_{i=1}^{k} \bar{N}_i D_i \bar{N}_i^t = I, \quad \operatorname{Ker} S_i \subset \operatorname{Im} Z_i.$$

The choice of D_i and Z_i will be specified later. Define the *coarse space*

$$\mathrm{W} = \{v \in V \ : \ v = \sum_{i=1}^{k} \bar{N}_i D_i u_i, u_i \in \operatorname{Im} Z_i\}.$$

Our algorithm is :

ALGORITHM 1. *Given $R \in V$, compute $U \in V$ as follows:*

(i) *Find λ_j so that $Z_i^t D_i^t \bar{N}_i^t \left(R - S\sum_{j=1}^k \bar{N}_j D_j Z_j \lambda_j\right) = 0$, $i = 1, \dots, k$.*
(ii) *Set $R_i = D_i^t \bar{N}_i^t (R - S\sum_{j=1}^k \bar{N}_j D_j Z_j \lambda_j)$.*
(iii) *Find a solution U_i for each of the local problems $S_i U_i = R_i$, $i = 1, \dots, k$.*
(iv) *Find μ_i so that $Z_i^t D_i^t \bar{N}_i^t \left(R - S\sum_{j=1}^k \bar{N}_j D_j (U_j + Z_j \mu_j)\right) = 0$, $i = 1, \dots, k$.*
(v) *The output is $U = \sum_{i=1}^k \bar{N}_i D_i (U_i + Z_i \mu_i)$.*

The solution of the auxiliary problem in step 1 can be omitted by choosing a suitable starting vector to guarantee that $\lambda_j = 0$ in every step. Mandel [**9**] has shown that the right-hand sides of the singular problems in step 3 are consistent, the output z from Algorithm 1 is independent of the choice of a solution in step 3, and the following condition number estimate holds.

THEOREM 1. *Algorithm* 1 *returns* $U = M^{-1}R$, *where* M *is symmetric positive definite and*

$$\kappa(M^{-1}S) = \lambda_{max}(M^{-1}S)/\lambda_{min}(M^{-1}S)$$

$$\leq \sup\left\{\frac{\sum_{j=1}^{k} \|\bar{N}_j^t \sum_{i=1}^{k} \bar{N}_i D_i U_i\|_{S_j}^2}{\sum_{i=1}^{k} \|U_i\|_{S_i}^2} : U_i \perp Ker(S_i),\ S_i U_i \perp Im\, Z_i\right\}. \tag{6}$$

The main trick in this paper is now to use the flexibility in the choice of the matrices Z_i to enforce that the supremum in (6) is taken only over vectors U_i such that the normal displacement $I_{\mathrm{HCT}} U_i$ is zero at all crosspoints. For this purpose, choose

$$Z_i = [X_1, \ldots, X_{n_i}, Y_{i1}, \ldots Y_{im_i}] \tag{7}$$

where $\{X_1, \ldots, X_{n_i}\}$, $n_i \leq 3$, is a basis of $\operatorname{Ker} S_i$, and for each crosspoint $j = 1, \ldots, m_j$ of Ω_i, Y_{ij} is a solution of the problems $S_i Y_{ij} = E_{ij}$, with E_{ij} the vector corresponding to a unit normal load applied at crosspoint j. Indeed, since S_i is symmetric, $S_i U_i \perp Y_{ij}$ implies that $U_i \perp S_i Y_{ij} = E_{ij}$, so $I_{\mathrm{HCT}} U_i$ is zero at all crosspoints.

It remains to construct the weight matrices D_i. If G is an edge or a crosspoint of Γ, define $E_G : V \to V$ as follows : $E_G(U)$ is the vector with the same values of the degrees of freedom as U on G, and zero values of all the other degrees of freedom. Here, an edge does not contain its end crosspoints, so $\sum_G E_G = I$. Now set, with $\beta \geq 1/2$,

$$D_i = \sum_{G \subset \partial\Omega_i} d(i,G) \bar{N}_i^t E_G \bar{N}_i, \qquad d(i,G) = \frac{\rho_i^\beta}{\sum\limits_{j: G \cap \partial\Omega_j \neq \emptyset} \rho_j^\beta}. \tag{8}$$

That is, the weight matrices D_i are diagonal, with the diagonal entry equal to the ratio of ρ_i^β to the sum of ρ_j^β for all subdomains sharing that degree of freedom. In our computations, we choose $\beta = 1$ as in [**5**].

4. Condition Number Estimate

The following theorem follows immediately from Theorem 3.3 in Mandel and Brezina [**10**].

THEOREM 2. *Let the weight matrices D_i be constructed as in (8) with $\beta \geq 1/2$, and for all subdomains crosspoints or edges $G \subset \partial\Omega_i \cap \partial\Omega_j$,*

$$(9) \qquad \frac{1}{\rho_j}\|\bar{N}_j^t E_G \bar{N}_i U_i\|_{S_j}^2 \leq \frac{1}{\rho_i} R \|U_i\|_{S_i}^2, \quad \forall U_i \perp Ker\, S_i, S_i U_i \perp Im\, Z_i .$$

Then the condition number from (6) satisfies $\kappa \leq 9(K+1)^2 R$, where K is the maximal number of adjacent subdomains to any subdomain Ω_i.

Define continuous analogues of the projection operators E_G via the interpolation mapping I_{HCT},

$$\mathcal{E}_G : \mathbb{H}_h(\Gamma) \to \mathbb{H}_h(\Gamma), \qquad \mathcal{E}_G I_{\mathrm{HCT}} U = I_{\mathrm{HCT}} E_G U, \qquad \forall U \in V.$$

Verification of the bound (9) will be based on estimates in the trace norm of the operators $\mathcal{E}_G$. For this purpose, we first state several technical results concerning the trace norm.

An extension lemma can be proved by similar arguments as in Widlund [**11**].

LEMMA 1. *For any $u \in \mathbb{H}_h(\partial\Omega_i)$, there is a $v \in \mathbb{H}_h(\Omega_i)$ so that $v|_{\partial\Omega_i} = u$, and $|\nabla v|_{1,\Omega_i} \leq C|\nabla u|_{1/2,2,\partial\Omega_i}$.*

From Lemma 1 and the trace inequality follows the equivalence of seminorms

$$(10) \qquad \frac{1}{\rho_i}|U|_{S_i}^2 \sim |\nabla I_{\mathrm{HCT}} U|_{1/2,2,\partial\Omega_i}^2 .$$

The following estimate of the trace norm of the extension by zero is proved as in Bramble, Pasciak, and Schatz [**2**, Lemma 3.5].

LEMMA 2. *There exists a constant C such that if the support of u is contained in a segment σ of $\partial\Omega_j$ of length τ, and $|\frac{\partial u}{\partial s}|_{0,\infty,\sigma} \leq \frac{c}{h}|u|_{0,\infty,\sigma}$, then*

$$|u|_{1/2,2,\partial\Omega_j}^2 \leq |u|_{1/2,2,\sigma}^2 + C\left(1 + \log\frac{\tau}{h}\right)|u|_{0,\infty,\sigma}^2 .$$

We will also need an extension of the discrete Sobolev inequality of Dryja [**6**] to piecewise polynomial functions of order $p > 1$.

LEMMA 3. *Let $p \geq 1$. Then there exists a constant $C = C(p)$ such that for every u continuous on $\partial\Omega_i$ such that $u \in P_p$ on the side of every triangle T,*

$$|\nabla u|_{0,\infty,\partial\Omega_i}^2 \leq C\left(1 + \log\frac{H}{h}\right)\left(|\nabla u|_{1/2,2,\partial\Omega_i}^2 + \frac{1}{H}|\nabla u|_{0,2,\partial\Omega_i}^2\right).$$

We are now ready for the main estimate.

LEMMA 4. *There exists a constant C such that if G is a crosspoint or an edge of Ω_i, then it holds for all $u \in \mathbb{H}_h(\Gamma)$, such that $u = 0$ on all crosspoints of Ω_i, that*

$$|\nabla \mathcal{E}_G u|_{1/2,2,\partial\Omega_j}^2 \leq C\left(1 + \log^\beta\frac{H}{h}\right)\left(|\nabla u|_{1/2,2,\partial\Omega_j}^2 + \frac{1}{H}|\nabla u|_{0,2,\partial\Omega_j}^2\right),$$

with $\beta = 1$ if G is a crosspoint and $\beta = 2$ if G is an edge.

PROOF. Assume $u \in \mathbb{H}_h(\Gamma)$ and $u = 0$ on all crosspoints of Ω_i. Let $F \in \partial\Omega_i$ be a crosspoint. The shape function $\phi_{\alpha,F}$ associated with the degree of freedom $\mathcal{E}_\alpha u(F)$ satisfies

$$(11) \quad |\nabla\phi_{\alpha,F}|_{1,2,\Omega_i} \le C, \qquad |\nabla\phi_{\alpha,F}|_{0,2,\Omega_i} \le Ch, \qquad |\nabla\phi_{\alpha,F}|_{0,\infty,\Omega_i} \le C.$$

From (11), the trace theorem, and Lemma 2 with $\tau = Ch$, it follows that

$$(12) \qquad |\nabla\phi_{\alpha,F}|_{1/2,2,\partial\Omega_i} \le C.$$

Since $u(F) = 0$, we have $\mathcal{E}_F u = \sum_\alpha \phi_{\alpha,F}\partial_\alpha u(F)$, and the proposition with $G = F$ follows using Lemma 3 and (12).

Let F_1, F_2 be crosspoints at the ends of an edge G. Since

$$\mathcal{E}_G u|_G = (u - \mathcal{E}_{F_1} u - \mathcal{E}_{F_2} u)|_G \ ,$$

it follows using the inequality $\|a + b\|^2 \le 2\left(\|a\|^2 + \|b\|^2\right)$, from the already proved estimate for the case of crosspoint, from inequalities (11), (12), and from Lemma 3, that

$$\begin{aligned}
|\nabla\mathcal{E}_G u|^2_{1/2,2,G} &\le C\left(1 + \log\frac{H}{h}\right)\left(|\nabla u|^2_{1/2,2,\partial\Omega_i} + \frac{1}{H}|\nabla u|^2_{0,2,\partial\Omega_i}\right)\\
|\nabla\mathcal{E}_G u|^2_{0,\infty,G} &\le C\left(1 + \log\frac{H}{h}\right)\left(|\nabla u|^2_{1/2,2,\partial\Omega_i} + \frac{1}{H}|\nabla u|^2_{0,2,\partial\Omega_i}\right)\\
|\nabla\mathcal{E}_G u|^2_{0,2,G} &\le |\nabla u|^2_{0,2,\partial\Omega_i} + Ch^2\left(1 + \log\frac{H}{h}\right)\left(|\nabla u|^2_{1/2,2,\partial\Omega_i} + \frac{1}{H}|\nabla u|^2_{0,2,\partial\Omega_i}\right).
\end{aligned}$$

Since $\mathcal{E}_G u = 0$ and $\nabla\mathcal{E}_G u = 0$ at F_1 and F_1, it remains only to apply Lemma 2 to $\partial_\alpha\mathcal{E}_G u$, $\alpha = 1, 2$. □

The desired bound on the condition number follows.

THEOREM 3. *Suppose that the assumptions made in Section 2 hold, that Z_i are defined by (7), and D_i are defined by (8). Then the condition number of Algorithm 1 satisfies*

$$\kappa \le C\left(1 + \log^2\frac{H}{h}\right),$$

with the constant C independent of H, h, and of the coefficients $\rho_i > 0$.

PROOF. The assumption (9) of Theorem 2 follows from Lemma 4, the equivalence of seminorms (10), and the inequality

$$|\nabla u|^2_{0,2,\partial\Omega_i} \le CH|\nabla u|^2_{1/2,2,\partial\Omega_i}$$

for all $u \in \mathbb{H}_h(\partial\Omega_i)$ that are zero at all crosspoints. □

TABLE 1. Results for a Rectangular Plate

	h	iter	condition
no corners	h	25	350
	$h/2$	29	1430
	$h/4$	32	5764
corners	h	11	5.4
	$h/2$	13	8.2
	$h/4$	14	11.8

2×8 substructures, regular decomposition
$\Omega = [-2, 2] \times [0, 20]$, 8×64 HCT elements for initial h

TABLE 2. Results for oval plate with 24 subdomains (Fig. 1)

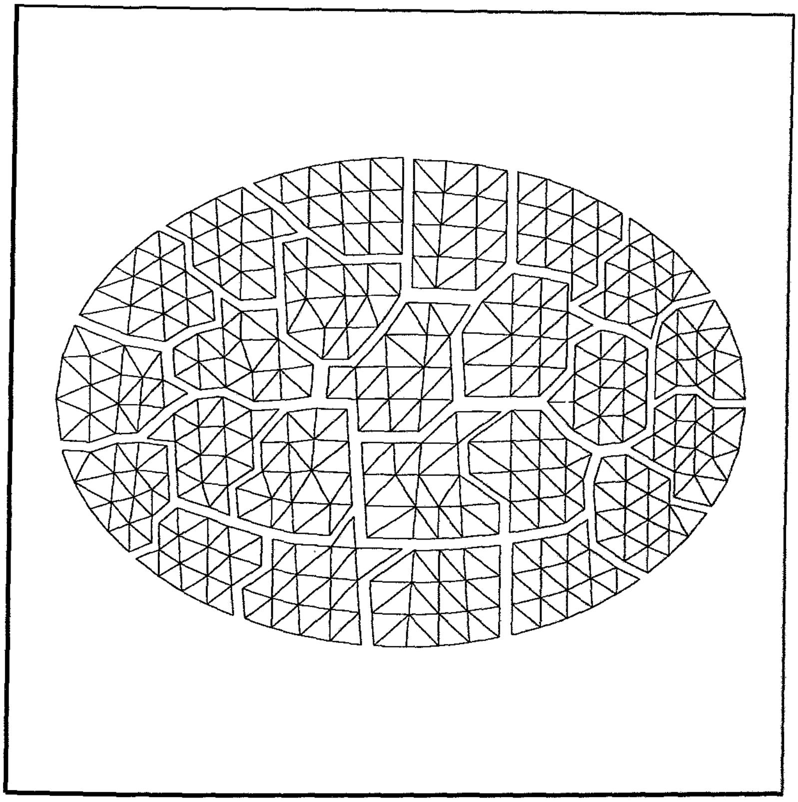
				CPU CRAY 2 sec	
	h	iter	cond	setup	iter
HCT element	h	43	153	14.8	3.6
no corners	$h/2$	59	588	25.8	7.6
	$h/4$	75	1981	53.9	27.7
HCT element	h	16	7.8	15.8	2.9
corners	$h/2$	23	22.2	26.0	4.3
	h/4	33	76.0	57.8	13.2
DKT element	h	33	62	14.9	3.2
no corners	$h/2$	49	239	25.1	6.5
	$h/4$	65	898	51.3	23.8
DKT element	h	12	3.3	15.4	2.6
corners	$h/2$	17	7.4	25.7	3.8
	$h/4$	25	25.1	56.6	10.8

5. Computational Results

In all tests, "corners" refers to the case when Z_i are defined by (7), and "no corners" is the case when the point load solutions Y_{ij} ("corner functions") are omitted from the columns of Z_i. The plate was clamped on the whole boundary. All experiments show that adding the corner functions improved the condition number considerably. The condition numbers were estimated from Ritz values in the Krylov space generated by conjugate gradients. The stopping criterion was the ratio of the ℓ^2 norm of the residual and the right hand side less than $\varepsilon = 10^{-6}$. In all experiments, the domain and the subdomains remain the same, and the elements are uniformly refined, so H is fixed. The condition number appears to grow about as $|\log^2 h|$ with the added corner functions, and about as $1/h^2$ without.

The purpose of the first test was to confirm the theory and demonstrate the effect of adding corner functions on the condition numbers (Tab. 1). Then to determine if adding the corner functions results in an improvement for

FIGURE 1. Oval plate



a realistic problem, we considered an oval plate discretized by an irregular mesh decomposed in 24 subdomains (Tab. 2, Fig. 1). The tests have shown that the improvement in the CPU times for the iterations outweigh the increase in the setup time due to the larger dimension of the coarse space.

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