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A Parallel Subspace Decomposition Method for Hyperbolic Equations

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ABSTRACT. A parallel subspace decomposition method for solving hyperbolic equations is presented. For a linear model problem, the conservation law is discretized by a cell vertex finite volume method on a triangular grid. Additive and multiplicative Schwarz methods together with a new sequence of subspace corrections are used for solving the normal equations, in a modification of the frequency decomposition approach [5]. Uniformly bounded convergence rates for all characteristic directions are obtained. The present approach may be extended to linear hyperbolic and elliptic systems [9].

1. Introduction

The numerical solution of flow problems is still a challenging task (see e. g. [1], [2], [6], [7], [10]). The governing equations, e. g. the Euler equations of steady flow, are a nonlinear system of composite elliptic/hyperbolic character. The efficient solution of hyperbolic equations is a prerequisite for fast solutions of the Euler equations.

Here we solve a simple hyperbolic model problem with constant coefficients:

$$a_1\,\partial_x u + a_2\,\partial_y u = 0\ ,\ a_2 > 0\ , \tag{1.1}$$

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This paper is in final form and no version of it will be submitted for publication elsewhere.

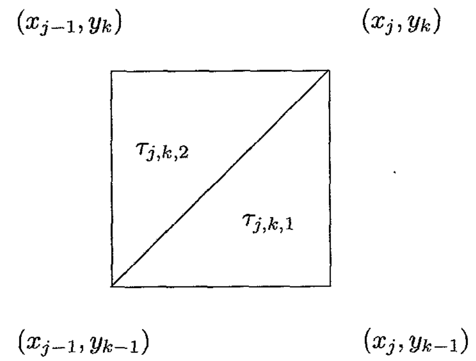

FIGURE 1. Subdivision of a square into two triangles

and x-periodic inflow and boundary conditions on the unit square. A conservative integral formulation is given by Stokes' theorem:

$$\int_{\partial\tau} a_1\, u\, dy - a_2\, u\, dx = 0\ , \tag{1.2}$$

for sufficiently smooth control elements τ with boundary $\partial\tau$.

A finite volume discretization on a triangular grid and a minimization problem are introduced in section 2. In section 3 we present a subspace decomposition approach and a parallel method based on an additive Schwarz iteration. Section 4 shows numerical results for additive and multiplicative Schwarz iteration. Details are given in [**8**] and [**9**].

2. Discrete minimization problem

A regular triangular grid is introduced by subdividing the cells of an equidistant grid according to Figure 1. This grid is equivalent to a triangular grid shown in Figure 2. The conservation equation (1.2) is approximated by the trapezoidal rule and yields for a triangle of type $\tau_{j,k,1}$:

$$Lu\,(\tau_{j,k,1}) = \frac{1}{h}\left(a_2\, u_{j,k} - (a_2 - a_1)u_{j,k-1} - a_1\, u_{j-1,k-1}\right) \tag{2.1}$$

and for a triangle of type $\tau_{j,k,2}$:

$$Lu\,(\tau_{j,k,2}) = \frac{1}{h}\left(a_1\, u_{j,k} + (a_2 - a_1)u_{j-1,k} - a_2\, u_{j-1,k-1}\right)\ . \tag{2.2}$$

As there are twice as many triangles as grid points, the discrete system $Lu = f$ is overspecified and no solution of the flux equations exists in the general case. Therefore a discrete minimization problem based on the squared Euclidean norm $E(u) = \|Lu - f\|_2^2$ is defined:

$$E(u^*) \le E(u)\ ,\ \forall\ u \in U\ . \tag{2.3}$$

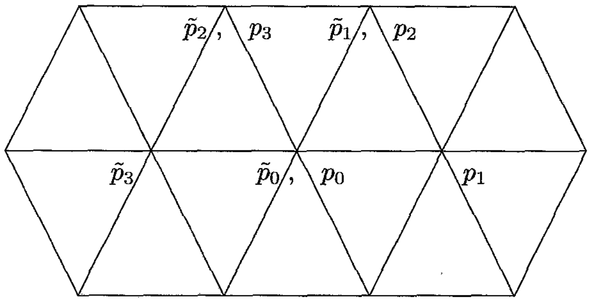

FIGURE 2. Location of coarse grid prolongations

The well known solution of this problem is given by the normal equations:

$$Au^* = L^*Lu^* = L^*f = b \; . \tag{2.4}$$

Matrix A is positive definite but not an M-matrix. The system is stable and the solution is unique. The discrete solution is second-order accurate on an equidistant grid (see [**8**]). This is rather surprising, because we do not fulfill the flux equations (2.1, 2.2) exactly but only in the mean. Apparently, the minimization procedure does not deteriorate the accuracy.

3. Subspace decomposition method

The discrete system (2.4) is solved by a subspace decomposition method. The additive Schwarz variant is perfectly suited for a parallel algorithm. We introduce several subspaces, U_κ = Range (p_κ), of the fine grid space U by prolongations $p_\kappa : V \to U$ on a coarse grid space V. The prolongations are then given in stencil notation:

$$p_0 = \frac{1}{2}\begin{bmatrix} 0 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \; p_1 = \frac{1}{2}\begin{bmatrix} 0 & 1 & -1 \\ -1 & 2 & -1 \\ -1 & 1 & 0 \end{bmatrix},$$

$$p_2 = \frac{1}{2}\begin{bmatrix} 0 & -1 & -1 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix}, \; p_3 = \frac{1}{2}\begin{bmatrix} 0 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 0 \end{bmatrix}.$$

Here, p_0 is the well known seven-point interpolation [**3**]. The other prolongations, defined on shifted coarse grids, are no more interpolations. The negative signs represent oscillating components transverse to selected characteristic directions. These prolongations are modifications of the frequency decomposition approach [**4, 5**]. As the stencil notation gives no information on the location of the coarse grid, Figure 2 shows the location of the center of the prolongations in the triangular grid.

Numerical tests showed that these four prolongations are not sufficient for a robust method. It was necessary to introduce four additional prolongations $\tilde{p}_0 \dots \tilde{p}_3$ with the same stencils as $p_0 \dots p_3$ but located at shifted coarse grid positions given in Figure 2.

On each subspace U_κ , $1 \leq \kappa \leq K$, smaller minimization problems are given by:

$$E(u + v_\kappa) \leq E(u + v) \ , \ \forall \ v \in U_\kappa \ . \tag{3.1}$$

This defines coarse grid corrections $G_\kappa(u) = u + v_\kappa$. We obtain the multiplicative Schwarz method:

$$\Phi^{MS} = \tilde{G}_3 \tilde{G}_2 \tilde{G}_1 \tilde{G}_0 G_3 G_2 G_1 G_0 \ . \tag{3.2}$$

With additional smoothing steps given by one iteration of the gradient method S, we obtain a twogrid method:

$$\Phi^{TG} = \tilde{G}_3 S \tilde{G}_2 S \tilde{G}_1 S \tilde{G}_0 S G_3 S G_2 S G_1 S G_0 S \ . \tag{3.3}$$

A parallel algorithm is obtained by an additive Schwarz method:

$$\Phi^{AS}(u) = u + \sum_{\kappa=1}^{K} \alpha_\kappa v_\kappa \ . \tag{3.4}$$

Where the coefficients α_κ are optimized by a small minimization problem:

$$E(\Phi^{AS}(u)) \leq E(u + v) \ , \ \forall \ v = \sum_{\kappa=1}^{K} \tilde{\alpha}_\kappa v_\kappa \ . \tag{3.5}$$

These corrections can be calculated in parallel on K processors and require the solution of coarse grid systems with different condition numbers. Standard iterative methods need different iteration counts on the processors which leads to load imbalance. Therefore a solution algorithm with time complexity independent of the condition, as e. g. a multigrid or a noniterative approach, is required for solving the coarse grid systems. The determination of the α_κ and the update of U is done sequentially on a single processor and needs communication. Although the solution of (3.5) is a small problem compared with the solution of all coarse grid corrections v_κ, it needs almost the same time as one correction. At the moment the small problem is solved sequentially which causes some load imbalance and reduces the parallel efficiency. The small problem should be solved parallel on several processors at the cost of increased communication. Results for parallel efficiencies are presented in [**9**].

TABLE 1. Convergence rates for subspace decomposition methods

q	-4.0	-2.0	-1.0	-0.5	0.0	0.25
$\rho(\Phi^{AS})$	0.87	0.88	0.88	0.87	0.74	0.87
$\rho(\Phi^{MS})$	0.63	0.64	0.64	0.64	0.47	0.65
$\rho(\Phi^{TG})$	0.64	0.54	0.55	0.53	0.44	0.50
q	0.50	0.75	1.0	1.5	2.0	4.0
$\rho(\Phi^{AS})$	0.88	0.86	0.73	0.88	0.87	0.86
$\rho(\Phi^{MS})$	0.67	0.65	0.47	0.63	0.64	0.62
$\rho(\Phi^{TG})$	0.50	0.52	0.44	0.54	0.56	0.59

4. Numerical results

The accuracy of the minimization solution is discussed in [**8**]. Here we present convergence rates for additive and multiplicative Schwarz iterations.

All results in Table 1 are asymptotic error reduction rates obtained on a 32×32 grid. The robustness of the two-grid iteration is analyzed. The characteristic direction, represented by the parameter $q = a_1/a_2$, has only minor influence on the convergence. Convergence rates are uniformly bounded and thus all presented methods are robust. As expected, the multiplicative Schwarz iteration is faster than the additive variant. With smoothing we obtain error reduction rates of approximately 0.5.

5. Conclusion

A new parallel algorithm for solving hyperbolic equations is presented. The linear advection equation is used as a model problem. The conservation form of the equations is discretized on a triangular grid by a cell vertex scheme. The overspecified system is transformed into a minimization problem which is uniformly stable for all characteristic directions. The solution is second order accurate on an equidistant grid.

The normal equations are solved with a subspace decomposition technique. Subspaces are defined by prolongations on a coarse grid. It is a modification of the frequency decomposition approach of Hackbusch [**4, 5**].

The multiplicative Schwarz iteration together with smoothing iterations shows good convergence rates of approximately 0.5. For the additive Schwarz iteration we obtain slower convergence rates of approximately 0.9, but the algorithm may easily be parallelized.

In all cases, the convergence rates are independent of the characteristic direction; thus the algorithm is robust. This is essential for future applications on flow problems with varying flow directions.

The present approach can be extended to linear and nonlinear systems. Results for linear systems are presented in [**9**]. The extension to Euler and Navier-Stokes equations is planned for future work.

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