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Interpolation Spaces and Optimal Multilevel Preconditioners

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ABSTRACT. This article throws light on the connection between the optimal condition number estimate for the BPX method and constructive approximation theory. We provide machinery which allows us to understand the optimality as a consequence of an approximation property and an inverse inequality in $H^{1+\epsilon}$, $\epsilon > 0$. This machinery constructs so-called *approximation spaces*, which characterize a certain rate of approximation by finite elements and relates them to interpolation spaces, which characterize a certain smoothness.

1. Introduction

For simplicity we consider the following elliptic boundary problem of second order on a polygonal domain $\Omega \subset \mathbb{R}^2$:

$$-\Delta u + u = f, \qquad \partial_n u|_{\partial\Omega} = 0,$$

where $f \in L^2(\Omega)$. The weak solution $u \in H^1(\Omega)$ is given by the variational problem

$$a(u,v) := (\nabla u, \nabla v)_{L^2} + (u,v)_{L^2} = (f,v)_{L^2} \qquad \forall v \in H^1(\Omega), \tag{1.1}$$

where we use a notation suggestive of the scalar products in $L^2(\Omega)^2$ and $L^2(\Omega)$. Let $\mathcal{T}_j$ be a sequence of nested regular quasi-uniform triangulations of Ω with mesh-size parameter

$$h_j := \max_{T \in \mathcal{T}_j} \operatorname{diam}(T) \approx 2^{-j}.$$

Throughout this article we use the notation $a \lesssim b$ iff there is a constant $c > 0$, such that $a \leq cb$ and $a \approx b$ iff $a \lesssim b$ and $a \gtrsim b$. The constants c will be independent of all parameters, except possibly of Ω and the shape regularity of the triangulations.

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Introducing the spaces of linear finite elements

$$X_j = \{u \in C(\bar{\Omega}) : u|_T \in P_1(T) \quad \forall T \in \mathcal{T}_j\},$$

where $P_1(T)$ denotes the linear functions on the triangle T, we get

$$X_0 \subset X_1 \subset \ldots \subset X_j \subset \ldots \subset H^1(\Omega).$$

The finite element operator $A_j : X_j \to X_j$ defined by

$$(A_j u, v_j)_{L^2} = a(u, v_j) \qquad \forall v_j \in X_j$$

should be *preconditioned* for efficient computation. Thus we seek for an easily invertible operator $B_j : X_j \to X_j$ such that $A_j \approx B_j$, i.e.,

$$(A_j u, u)_{L^2} \approx (B_j u, u)_{L^2} \qquad \forall u \in X_j.$$

Bramble, Pasciak and Xu [**4, 14**] constructed the preconditioner

$$B_j = A_0 Q_0 + \sum_{k=1}^{j} 4^k (Q_k - Q_{k-1}),$$

where $Q_k : L^2(\Omega) \to X_k$ are the L^2-orthogonal projections. They were originally able to prove without any regularity assumption on the problem (1.1)

$$\frac{1}{j+1}(B_j u, u)_{L^2} \lesssim (A_j u, u)_{L^2} \lesssim (j+1)\,(B_j u, u)_{L^2} \qquad \forall u \in X_j. \tag{1.2}$$

Their proof was based on the observation that

$$(A_k u, u)_{L^2} \approx 4^k \|u\|_{L^2}^2 \qquad \forall u \in \mathrm{range}(Q_k - Q_{k-1}),$$

which is a fairly easy consequence of the *approximation property*

$$\|u - Q_k u\|_{L^2} \lesssim h_k \|u\|_{H^1} \qquad \forall u \in H^1(\Omega) \tag{1.3}$$

and the *inverse inequality*

$$\|u_k\|_{H^1} \lesssim h_k^{-1} \|u_k\|_{L^2} \qquad \forall u_k \in X_k. \tag{1.4}$$

In the case of $H^{1+\alpha}$-regularity of the problem (1.1), $1/2 \le \alpha \le 1$, Xu [**4, 14**] was able to improve the factor $1/(j+1)$ of the lower bound in (1.2) to $(j+1)^{1-1/\alpha}$.

Oswald [**10**] was the first to observe a strong link of this method of preconditioning to approximation theory, a link which, in fact, *immediately* supplies a proof for the optimal result

$$A_j \approx B_j, \tag{1.5}$$

without any regularity assumption on the problem (1.1). Several authors have subsequently added generalizations or constructed more or less elementary proofs [**2, 3, 7, 15, 16**]. The aim of this article is to clarify the link to approximation theory by making available an easily accessible framework. Moreover it will turn out that the main ingredients of the proof are inequalities like (1.3) and (1.4).

2. Approximation Spaces are Interpolation Spaces

The optimality result (1.5) would be a straightforward consequence of the following norm equivalence with *scaling exponent* $\theta = 1$:

$$\|u\|_{H^1}^2 \approx \|u\|_{L^2}^2 + \sum_{k=0}^{\infty} \left(2^{k\theta} E_k(u)\right)^2 \qquad \forall u \in H^1(\Omega), \tag{2.1}$$

where $E_k(u)$ denotes the *error of best approximation* in $L^2(\Omega)$

$$E_k(u) = \inf_{v_k \in X_k} \|u - v_k\|_{L^2} = \|u - Q_k u\|_{L^2}.$$

We now ask the rather abstract question: Which sequences X_j of nested finite-dimensional subspaces of $L^2(\Omega)$ allow for some scaling exponent θ such that the norm equivalence (2.1) holds?

Rather than answering this question directly, we define Banach spaces $A^\theta \hookrightarrow L^2(\Omega)$ by the norms given as the right hand sides of (2.1),

$$\|u\|_{A^\theta}^2 = \|u\|_{L^2}^2 + \sum_{k=0}^{\infty} \left(2^{k\theta} E_k(u)\right)^2.$$

These *approximation spaces* A^θ, which measure by θ how well their elements can be approximated by the spaces X_j, were introduced by Peetre [**6, 11**] in the early sixties and have been intensively studied in approximation theory since then. It should be mentioned that the results to follow were known in a somewhat different form to the Russian school around Nikol'skiĭ as early as the fifties.

Our starting question reads now: Is there a θ such that $A^\theta = H^1(\Omega)$? This question is a key issue of approximation theory — it requires the characterization of the approximation spaces through *smoothness spaces* like the Sobolev spaces. The answer given by Peetre [**6, 11**] was a relation between the approximation spaces A^θ and the *interpolation spaces* $(L^2(\Omega), X)_{\sigma,2}$, where X is some "nice" space with $X_k \subset X \subset L^2(\Omega)$ for all $k \geq 0$.

THEOREM 1. *Let $\alpha > 0$. An approximation property (Jackson inequality) J_α, i.e.,*

$$\|u - Q_k u\|_{L^2} \lesssim 2^{-k\alpha} \|u\|_X \qquad \forall k \geq 0, u \in X, \tag{2.2}$$

implies the embedding

$$\left(L^2(\Omega), X\right)_{\sigma,2} \hookrightarrow A^{\sigma\alpha} \qquad 0 < \sigma < 1.$$

Let $\beta > 0$. An inverse inequality (Bernstein inequality) B_β, i.e.,

$$\|u_k\|_X \lesssim 2^{k\beta} \|u_k\|_{L^2} \qquad \forall k \geq 0, u_k \in X_k, \tag{2.3}$$

implies the embedding

$$A^{\sigma\beta} \hookrightarrow \left(L^2(\Omega), X\right)_{\sigma,2} \qquad 0 < \sigma < 1.$$

REMARK 1. Note that a standard interpolation argument applied to the Jackson inequality J_α for X and to the trivial estimate $\|u - Q_k u\|_{L^2} \lesssim \|u\|_{L^2}$ would only reveal an approximation property for the interpolation spaces

$$2^{k\alpha\sigma}\|u - Q_k u\|_{L^2} \lesssim \|u\|_{(L^2(\Omega),X)_{\sigma,2}} \qquad 0 < \sigma < 1. \tag{2.4}$$

This result would be considerably weaker than the assertion of Theorem 1, which states that the right hand sides of (2.4) are in fact square summable as a sequence of k.

If we use an appropriate method for the construction of the interpolation spaces $(L^2(\Omega), X)_{\sigma,2}$, the proof of Theorem 1 is simple. We demonstrate this for the first part concerning the Jackson inequality.

PROOF. Fix some $0 < \lambda < 1$. Using the discrete version of Peetre's K-method of interpolation **[1, 6, 13]** we get

$$\|u\|^2_{(L^2(\Omega),X)_{\sigma,2}} = \sum_{k\in\mathbb{Z}} \left(\lambda^{-k\sigma} K(\lambda^k, u)\right)^2.$$

The following estimates relate the *K-functional* with the error of best approximation by using the Jackson inequality J_α: For all $k \geq 0$

$$\begin{aligned} E_k(u) &\leq \inf_{v\in X} \|u - Q_k v\|_{L^2} \\ &\leq \inf_{v\in X} \left(\|u - v\|_{L^2} + \|v - Q_k v\|_{L^2}\right) \\ &\lesssim \inf_{v\in X} \left(\|u - v\|_{L^2} + 2^{-k\alpha}\|v\|_X\right) =: K(2^{-k\alpha}, u). \end{aligned}$$

Thus, by making the choice $\lambda = 2^{-\alpha}$, we end up with

$$\|u\|^2_{(L^2(\Omega),X)_{\sigma,2}} \gtrsim \sum_{k=0}^{\infty} \left(2^{k\sigma\alpha} E_k(u)\right)^2 + \|u\|^2_{L^2} = \|u\|^2_{A^{\sigma\alpha}}.$$

□

Let us note that the discrete version of Peetre's J-method of interpolation **[1, 6, 13]** turns out to be appropriate for the proof of the second part of the Theorem.

COROLLARY 1. *The approximation property is restricted by the inverse inequality, i.e.,* J_α *and* B_β *imply* $\alpha \leq \beta$. *For* $\alpha = \beta$ *we get the identification*

$$A^{\sigma\alpha} = \left(L^2(\Omega), X\right)_{\sigma,2}, \qquad 0 < \sigma < 1.$$

PROOF. If J_α and B_β hold, Theorem 1 gives $A^{\sigma\beta} \hookrightarrow A^{\sigma\alpha}$ for $0 < \sigma < 1$. This embedding is equivalent to $\alpha \leq \beta$, as can easily be shown. □

3. Application to Linear Finite Elements

In order to answer the question from the beginning of our consideration, Corollary 1 leads us to the following strategy: Choose X and $0 < \sigma < 1$, such that

$$H^1(\Omega) = \Big(L^2(\Omega), X\Big)_{\sigma,2}.$$

In any case this requires that X is *smoother* than $H^1(\Omega)$. Interpolation theory in Sobolev spaces [**13, 1**] states for *minimally smooth* domains Ω (i.e., Ω allows a continuous extension operator $E : H^s(\Omega) \to H^s(\mathbb{R}^2)$ for all $s \geq 0$), that

$$H^1(\Omega) = \Big(L^2(\Omega), H^s(\Omega)\Big)_{1/s,2} \qquad \forall s > 1.$$

In our context it suffices to know, that a polygonal domain Ω *without slits* is minimally smooth [**12**]. Now it turns out, that the finite element spaces fulfill

$$X_k \subset H^s(\Omega) \qquad \Longleftrightarrow \qquad 0 \leq s < 3/2.$$

For the following we fix some $1 < s = 1 + \epsilon < 3/2$ and we can apply Theorem 1 — as soon as we have established an approximation property and an inverse inequality in $H^{1+\epsilon}$. We obtain the approximation property J_s, i.e.,

$$\|u - Q_k u\|_{L^2} \lesssim h_k^s \|u\|_{H^s} \lesssim 2^{-ks}\|u\|_{H^s} \qquad \forall u \in H^s(\Omega), \tag{3.1}$$

by simple interpolation between the cases $s = 0$ and $s = 2$, as indicated in Remark 1. A little bit deeper lies the inverse inequality B_s, i.e.,

$$\|u_k\|_{H^s} \lesssim h_k^{-s}\|u_k\|_{L^2} \lesssim 2^{ks}\|u_k\|_{L^2} \qquad \forall u_k \in X_k, \tag{3.2}$$

which can be proved using the Sobolev-Slobodeckiĭ norm

$$\|u\|^2_{H^{1+\epsilon}} \approx \|u\|^2_{H^1} + \int_\Omega\int_\Omega \frac{|\nabla u(x) - \nabla u(y)|^2}{|x-y|^{2+2\epsilon}}\, dx\, dy$$

of $H^s(\Omega)$, cf. [**5**]. Thus we have $\alpha = \beta = 1/\sigma$ and Corollary 1 states the equivalence, which makes the BPX preconditioner optimal:

THEOREM 2. *For linear finite elements the equivalence $A^1 = H^1(\Omega)$ holds.*

REMARK 2. This Theorem and the more general equivalence "approximation space = smoothness space", i.e, $A^s = H^s(\Omega)$ for s from some interval, holds generically for a lot of sequences X_j, like higher order finite elements, spectral methods and wavelets. Details can be found in [**8, 9**].

Our considerations show that it is reasonable to view the approximation property (3.1) and the inverse inequality (3.2) in $H^{1+\epsilon}(\Omega)$, $\epsilon > 0$ arbitrarily small, as the "chief cause" for the optimality of BPX. In the original proof of the weaker result (1.2) corresponding estimates in $H^1(\Omega)$ were the groundwork. Thus, the essential step is to use the fact that linear finite elements are a little bit smoother than one usually thinks. This essential step is hidden in one way or another in all proofs [**2, 3, 7, 10, 15, 16**] of the optimality of the BPX preconditioner.

REMARK 3. The reader should not confuse the concept of regularity $X_j \subset H^{1+\epsilon}$ of the approximating spaces X_j with the concept of $H^{1+\alpha}$-regularity of the problem (1.1). The first is a *general* property of the *method* of approximation, the second holds only for *special* cases of the underlying *problem.*

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