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A Spectral Stokes Solver in Domain Decomposition Methods

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ABSTRACT. We present a domain decomposition solver for the linear Stokes problem. We use a technique based on an equivalence between the single-domain and multidomain formulations of the Stokes problem in which the transmission conditions at subdomain interfaces are properly taken in account. The discrete problem is solved using the Uzawa algorithm.

1. Introduction

The aim of this note is to present a domain decomposition method for the spectral solution of Stokes equations. After recalling the spectral collocation method in its single–domain version, we introduce our multidomain approach which is based on the concept of transmission interface conditions. The resulting problem is handled by an Uzawa solution algorithm which requires at each iteration the solution of a Poisson boundary value problem for each velocity component. At this stage we apply the projection decomposition method introduced in [5]. We conclude by a numerical investigation that shows the efficiency and accuracy of our approach.

2. The Stokes Problem

The Stokes problem reads : find u and p such that

$$
\begin{aligned}
-\nu\Delta u+\nabla p &= f \quad \text{in} \quad \Omega\,,\\
\nabla\cdot u &= 0 \quad \text{in} \quad \Omega\,,\\
u &= 0 \quad \text{on} \quad \partial\Omega\,,
\end{aligned}
\tag{1}
$$

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where $\Omega =]-1,1[^2$, f is the density force and ν the viscosity. This problem admits the following variational formulation:

Find $u \in X$ and $p \in M$ such that

$$\nu(\nabla u, \nabla v) - (\nabla \cdot v, p) = < f, v >, \ \forall \ v \in X \tag{2}$$

$$(\nabla \cdot u, q) = 0 \ \forall \ q \in M, \tag{3}$$

where $X = \left(H_0^1(\Omega)\right)^2$ and $M = L_0^2(\Omega) = \left\{v \in L^2(\Omega), \int_\Omega v dx = 0\right\}$. Here $(\cdot,\cdot)$ denotes the scalar product of $L^2(\Omega)$ and $< \cdot,\cdot >$ the duality between X' and X (for notation see, e.g., Lions and Magenes [1]).

This problem is well posed and has a unique solution in $X \times M$.

3. Spectral single-domain discretization

We need first to introduce the polynomial spaces of approximation. For the velocity we choose $X_N = (P_N^0(\Omega))^2$ and for the pressure $M_N = P_N(\Omega) \bigcap L_0^2(\Omega)$, where $P_N(\Omega)$ denotes the space of algebraic polynomials of degree not greater than N with respect to each coordinate.

Let L_N be the Legendre polynomial of degree N, the solution of the Sturm-Liouville equation

$$\left((1-x^2)L_N'(x)\right)' + N(N+1)L_N(x) = 0, \quad -1 < x < 1.$$

We introduce the following discrete inner product:

$$(u,v)_N = \sum_{i,j=0}^{N} u(\xi_i,\xi_j)v(\xi_i,\xi_j)\rho_i\rho_j, \quad \forall (u,v) \in (C^0(\bar\Omega))^2, \tag{4}$$

where $\{\xi_j, 0 \le j \le N\}$ are the $N+1$ Gauss-Lobatto-Legendre points, the roots of the polynomial: $(1-x^2)L_N'(x)$, and $\{\rho_j, 0 \le j \le N\}$ are the associated weights (see, e.g., [2]). Then the approximate problem based on the Legendre collocation method reads : Find (u_N, p_N) in $X_N \times M_N$ such that for all $v_N \in X_N$ and all $q_N \in M_N$

$$\mathcal{A}_N(u_N, v_N) + b_N(v_N, p_N) = (f, v_N)_N, \tag{5}$$

$$b_N(u_N, q_N) = 0, \tag{6}$$

where $\mathcal{A}_N(u,v) = \nu(\nabla u, \nabla v)_N$ and $b_N(v,q) = -(\nabla . v, q)_N$.

Both velocity and pressure will be computed on the Gauss-Lobatto grid Ξ_N:

$$\Xi_N = \{x = (\xi_i, \xi_j), 0 \le i \le N, 0 \le j \le N\}.$$

As proven by Bernardi and Maday [3], the drawback of the previous formulation is that it is affected by spurious modes. These are nonzero polynomials $q^* \in P_N(\Omega) \bigcap L_0^2(\Omega)$ such that $b_N(v, q^*) = 0$, $\forall v \in X_N$. If such a q^* exists then each couple $(u_N, p_N + q^*)$ would also be a solution, yielding a potential instability in the pressure approximation.

The subspace $\mathcal{Z}_N$ of $P_N(\Omega)\bigcap L_0^2(\Omega)$ made of spurious modes has dimension 7 and is spanned by

$$L_N(y),\ L_N(x),\ L_N(x)L_N(y), L_N'(x)L_N'(y),$$
$$L_N'(x)yL_N'(y),\ xL_N'(x)L_N'(y),\ xL_N'(x)yL_N'(y).$$

To get rid of these undesirable modes and to get satisfactory approximation, the pressure space M_N^o can be chosen such that it is a supplementary space of $\mathcal{Z}_N$, i.e.

$$P_N(\Omega)\cap L_0^2(\Omega) = M_N \oplus \ \mathcal{Z}_N. \tag{7}$$

In this case, one can prove the existence of an Inf-Sup condition with a constant behaving as $\mathcal{O}(N^{-1})$ [3], i.e.,

$$\inf_{q\in M_N^o}\ \sup_{v\in X_N} \frac{b_N(v,q)}{\|v\|_X\|q\|_M} = \mathcal{O}(N^{-1}).$$

The discrete problem $(5)-(6)$ can be solved using the following Uzawa algorithm [2]. Formally we write

$$u_N = S_N^{-1}\left(f - \nabla_N p_N\right) \tag{8}$$

where S_N and ∇_N are the algebraic representations of $\mathcal{A}_N$ and b_N, respectively. Inserting u_N in the continuity equation we get:

$$\nabla_N.S_N^{-1}\nabla_N p_N = \nabla_N.S_N^{-1}f. \tag{9}$$

The Uzawa operator is then

$$\nabla_N.S_N^{-1}\nabla_N \tag{10}$$

and it is positive, symmetric with a condition number of $O(N^2)$. This enables the use of conjugate gradient iterations to solve (9).

4. Non-overlapping domain decomposition

Our non-overlapping domain decomposition solver will be based on Uzawa's algorithm.

The solution of such a system is accomplished via a global inner/outer iterative scheme where at each iteration only elliptic problems need to be solved. An equivalence principle between single and multidomain formulations of the Helmholtz problem is used in this case, and the same iterative algorithm is applied to the resolution of the inner iteration yielding now a sequence of single domain elliptic problems.

Let Ω be a bounded domain (that we assume to be a rectangle for the sake of simplicity) of $\mathbb{R}^2$ partitioned into Ω_1 and Ω_2 with $\Gamma = \partial\Omega_1\bigcap\partial\Omega_2$.

For the velocity space we take

$$X_h = \left\{v_h \in \left(\mathcal{C}^0(\bar\Omega)\right)^2 \ \mid v_h^i = v_h|_{\Omega_i} \in \left(P_N(\Omega_i)\right)^2, v_h|_{\partial\Omega} = 0\right\}. \tag{11}$$

and for the pressure

$$M_h = \left\{ q_h \in \mathcal{C}^0(\bar{\Omega}) \mid q_h^i = q_h|_{\Omega_i} \in P_N(\Omega_i), \int_\Omega q_h = 0 \right\}. \tag{12}$$

Note that the pressure is required to be continuous over Ω a priori.

The multidomain problem is defined as follows.

Find (u_h, p_h) in $X_h \times M_h$ such that for all v_h in X_h and for all q_h in M_h

$$\mathcal{A}_h(u_h, v_h) + b_h(v_h, p_h) = (f, v_h)_h \tag{13}$$

$$b_h(u_h, q_h) = 0, \tag{14}$$

where $\mathcal{A}_h(u_h, v_h) = \sum_{i=1,2} \nu \left(\nabla u_h^i, \nabla v_h^i\right)_{N,i}$, $b_h(v_h, q_h) = -\sum_{i=1,2} \left(\nabla . u_h^i, q_h^i\right)_{N,i}$, and where $(.,.)_{N,i}$ denotes the generalization of the discrete inner product $(.,.)_N$ on Ω_i based on transformed nodes ξ_j^i and weights ρ_j^i. Also in this case the problem is well-posed when the pressure is taken in a space complementary to the one of the spurious modes. Concerning the latter, taking for instance the domain $\Omega =]-2,2[\times]0,2[$, it can be proven (see [4]) that the space of spurious modes

$$\mathcal{Z}_h = \{q_h \in M_h : b_h(v, q_h) = 0 \, \forall v \in X_h\},$$

has dimension 7. A set of generators is provided by the following three functions

$$\begin{aligned} &(L_N(x+1), (-1)^N L_N(x-1)), \\ &(L_N(y), L_N(y)), \\ &(L_N(x+1)L_N(y), (-1)^N L_N(x-1)L_N(y)) \end{aligned} \tag{15}$$

(for each couple, the first component denotes the value attained in Ω_1, the second one is in Ω_2) plus four other functions which are the characteristic polynomials associated with the four corner points of Ω. For a domain Ω of more general shape (i.e., made by a union of rectangles), we still have the three spurious modes (15), plus as many characteristic functions as the number of corners of $\partial\Omega$. We point out that the existence of internal cross points does not introduce additional spurious modes. The latter property would not be true if we used a space of discountinuous pressures.

The pressure subspace M_h^0 that we are going to use is a complementary space of $\mathcal{Z}_h$, i.e. it satisfies $M_h = M_h^0 \oplus \mathcal{Z}_h$, and we propose a subdomain iteration method to solve the discrete problem.

5. A solver based on subdomain iterations

Using the spectral collocation method, the formulation $(13)-(14)$ of the multidomain method can be given the following pointwise interpretation:

Find $(u_h, p_h) \in X_h \times M_h$ such that

$$\begin{aligned} -\nu\Delta u_h^1 + \nabla p_h^1 = f_h^1 \quad & \text{in } \Xi_N^1 \cap \Omega_1 \\ \nabla . u_h^1 = 0, \quad & \text{in } \Xi_N^1 \backslash \Gamma \\ u_h^1 = 0 \quad & \text{on } \Xi_N^1 \cap \partial\Omega; \end{aligned}$$

and

$$\begin{aligned} -\nu\Delta u_h^2 + \nabla p_h^2 = f_h^2, &\quad \text{in } \Xi_N^2 \bigcap \Omega_2 \\ \nabla . u_h^2 = 0, &\quad \text{in } \Xi_N^2 \backslash \Gamma \\ u_h^2 = 0 &\quad \text{on } \Xi_N^2 \bigcap \partial\Omega. \end{aligned}$$

The interface conditions (according to [6]) are:

$$u_h^1 = u_h^2 \quad \text{on} \quad \Xi_N^\Gamma, \tag{16}$$

$$\left(\nu \frac{\partial u_h^2}{\partial n_\Gamma} - p_h^2 n_\Gamma\right) - \left(\nu \frac{\partial u_h^1}{\partial n_\Gamma} - p_h^1 n_\Gamma\right) = -w^2 R_h^2 - w^1 R_h^1 \quad \text{on} \quad \Xi_N^\Gamma, \tag{17}$$

$$p_h^1 = p_h^2 \quad \text{on} \quad \Xi_N^\Gamma, \tag{18}$$

$$\omega^1 \nabla \cdot u_h^1 + \omega^2 \nabla \cdot u_h^2 = 0 \quad \text{on} \quad \Xi_N^\Gamma. \tag{19}$$

Here, n_Γ is the normal unit vector on Γ directed outward with respect to Ω_2, $\omega^1 = \rho_N^1$ and $\omega^2 = \rho_0^2$, $R_h^i = -\nu\Delta u_h^i + \nabla p_h^i - f_h^i$, $i = 1, 2$, and Ξ_N^i is the set of Legendre Gauss-Lobatto nodes in $\bar{\Omega}_i$ while

$$\Xi_N^\Gamma = \Xi_N^1 \bigcap \Gamma = \Xi_N^2 \bigcap \Gamma.$$

Conditions (16) yield the continuity of the velocity, whereas (17) and (18) enforce the continuity of the normal derivative in a weak form, i.e.

$$\nu \frac{\partial u_h^2}{\partial n_\Gamma} - \nu \frac{\partial u_h^1}{\partial n_\Gamma} = -w^2 R_h^2 - w^1 R_h^1 \ \ on \ \ \Xi_N^\Gamma. \tag{20}$$

This global system is solved by using the Uzawa algorithm, consisting of deducing a global problem for the pressure field (p_h^1, p_h^2) similar to (9) and then solving it by a conjugate gradient procedure. At any step one Poisson problem for each velocity component is solved. At this stage we apply a domain decomposition technique which enforces both (16) and (20) on Γ. Among all available domain decomposition spectral solvers, we apply the one based on the Spectral Projection Method (see [5]). This algorithm is effective and allows spectral accuracy to be achieved.

6. Some numerical results

Our test case concerns the approximation of (1) corresponding to the exact solution: $u_{ex} = (\sin(\pi x)\sin(\pi y), \sin(\pi x)\sin(\pi y))$ and $p_{ex} = \sin(\pi x)\cos(\pi y)$, defined on the domain $\Omega =]0,1[^2$.

Let $\|u_h\|_h = \left(\sum_{i=1}^{ndom} (\nabla . u_h, \nabla . u_h)_{N,i}\right)^{\frac{1}{2}}$ be the discrete norm and *ndom* is the number of subdomains (of equal measure) into which the domain Ω has been subdivided.

We recall that the numerical divergence is zero at all the collocation nodes of $\bar{\Omega}_1$ and $\bar{\Omega}_2$, except those lying on the interface Γ where, however, (19) holds.

We are also interested in the number of Uzawa iterations using a suitable mass diagonal matrix preconditioner (see [7]). We report the number of Uzawa iterations, for several numbers of subdomains, taking the polynomial degree N equal to 8 within each subdomain (Table 1)

Table 1

Subdomains	1x1	2x2	3x3	4x4	5x5	6x6	7x7	8x8	9x9	10x10
Iterations	7	16	16	16	18	18	16	16	16	16

We observe that the iteration number is independent of the number of subdomains. In Table 2 we report the value of the divergence norm $\|u_h\|_h$ for several number of subdomains (left) and different value of the polynomial degree N on each subdomain (top). Spectral approximation is verified.

Table 2

	6	8	10	12	14	16
1x1	$0.2e-1$	$0.5e-3$	$0.4e-5$	$0.3e-7$	$0.1e-9$	$0.9e-12$
2x2	$0.3e-3$	$0.3e-5$	$0.7e-7$	$0.3e-10$	$0.2e-12$	$0.2e-11$
4x4	$0.3e-4$	$0.3e-7$	$0.2e-10$	$0.1e-11$	$0.4e-12$	$0.4e-11$
8x8	$0.9e-6$	$0.2e-9$	$0.8e-12$	$0.2e-11$	$0.9e-11$	$0.8e-11$
10x10	$0.3e-7$	$0.5e-10$	$0.1e-11$	$0.3e-11$	$0.8e-11$	$0.9e-11$

In these tables, we have indeed used the full space M_h defined in (12). The spurious modes that are intrinsically produced by our scheme (13) – (14) are easy to filter out along the Uzawa iteration process and the pressure obtained is then filtered by using the method described in [8]

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