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A χ-Formulation of the Viscous-Inviscid Domain Decomposition for the Euler/Navier-Stokes Equations

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ABSTRACT. In this paper we present an application of the χ-formulation for the solution of the Navier-Stokes equations. Subsonic laminar and transonic turbulent flows are calculated.

1. Introduction

Many flow phenomena, such as turbulent flows, involve a wide range of length scales, for this reason their numerical simulation is a challenging problem. In many situations, high accuracy is necessary only in limited parts of the domain, one possibility is to resolve the physics on a global uniform grid with the smallest desired mesh size. However this direct approach is far from being efficient. For these reasons several physically motivated domain decomposition methods have been proposed. In this paper, we present a domain decomposition method termed χ-formulation [3]. The key idea is to locally evaluate the magnitude of the diffusive part of the Navier-Stokes equations, and by this inspection to detect the smallest scales involved into the phenomenon under investigation. In this way, we obtain an automatic detection of the shear layers (such as boundary layers, wakes, etc...) where an highly accurate simulation is required, and a natural splitting of the domain can be performed.

The χ-formulation has already been successfully applied to a scalar model problem [2], showing its ability in optimizing the interface position. In the present paper, we present an application of this approach to the solution of the compressible Navier-Stokes equations for two-dimensional subsonic and transonic flows, in laminar and turbulent regimes.

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2. The χ-Formulation for the Navier-Stokes Equations

Consider the incompletely parabolic problem

$$(2.1) \qquad \begin{cases} u_t + Bu_x + Cu = Au_{xx} \ , \quad x \in (0, L) \ , \ t > 0 \, , \\ + \textit{ initial and boundary conditions} \ , \end{cases}$$

where $u = \binom{u_1}{u_2}$, $A = \left(\begin{smallmatrix} 0 & 0 \\ 0 & \nu \end{smallmatrix}\right)$ with $\nu > 0$, and B a symmetric matrix. The χ-formulation replaces problem (2.1) by a modified problem, in which the diffusive term is deleted when it is negligible. To this end, let us choose a cut-off parameter $\delta > 0$, and a further parameter $\sigma > 0$ such that $\sigma \ll \delta$, and let us introduce the monotone function $\chi = \mathbb{R} \to \mathbb{R}$ defined as:

$$(2.2) \qquad \begin{cases} \chi(s) = \begin{cases} s & \text{if } s > \delta \ , \\ (s - \delta + \sigma)(\delta/\sigma) & \text{if } \delta - \sigma \le s \le \delta \ , \\ 0 & \text{if } 0 \le s < \delta - \sigma \ , \end{cases} \\ \chi(s) = -\chi(-s) \ , \quad s < 0 \ . \end{cases}$$

We define the *χ-formulation* of problem (2.1), as the following modified problem

$$(2.3) \qquad \begin{cases} u_t + Bu_x + Cu = A\chi(u_{xx}) \ , \quad x \in (0, L) \ , \ t > 0 \, , \\ + \textit{ initial and boundary conditions} \ , \end{cases}$$

where the viscous term $Au_{xx} = (0, \nu u_{2,xx})^T$ is replaced by the modified term $(0, \nu\chi(u_{2,xx}))^T$, denoted by $A\chi(u_{xx})$. The linear ramp of size σ in the function χ yields a continuous transition between the state $\chi = 0$ (the inviscid state) and states $| \chi | \ge \delta$ (the fully viscous region). This formulation leads to a smooth behavior of the solution u of (2.3) at the viscous/inviscid interface.

In [**3**], it has been proved that the χ-formulation leads to a well posed problem for the scalar advection-diffusion problem, and that the maximun deviation of u from the solution of (2.1) is proportional to $\delta\nu$, providing an estimate for the choice of the parameter δ as a function of the diffusion parameter ν. Moreover it has been shown that u is continuously differentiable all over the domain, in particular across the interface. This property is peculiar to the χ-formulation. Indeed any a priori choice of the viscous/inviscid interface leads only to C^0 continuity. Further in [4] the same results have been proved for more general boundary conditions, and in [**5**] the convergence of the iterative self-adaptive domain decomposition has been investigated.

The χ-formulation introduces a nonlinearity into the diffusive part of the equations. However, if the original problem is already nonlinear, this added nonlinearity can be treated explicitly as the other nonlinearities. In the case of the viscous Burgers equation, it has been shown [**2**] that the χ-Burgers equation can be solved with a computational effort which is comparable with the cost of solving the Burgers equation on the same domain. In [**1**], Achdou and Pironneau have extended the χ-formulation to the incompressible Navier-Stokes equation, showing how the self-adaptive detection of the viscous subdomain greatly improve the efficiency of the simulations.

In this work we present an extension of [**3**] to the compressible Navier-Stokes equations, which for two-dimensional compressible flows, can be written in the nondimensional form

$$\frac{\partial Q}{\partial t} + \frac{\partial F}{\partial x} + \frac{\partial G}{\partial y} = \frac{1}{Re}\left(\frac{\partial F_v}{\partial x} + \frac{\partial G_v}{\partial y}\right) \quad , \tag{2.4}$$

where $Q = (\rho, \rho u, \rho v, E)^T$,

$$F = \begin{pmatrix} \rho u \\ \rho u^2 + p \\ \rho uv \\ (E+p)u \end{pmatrix} \quad , \quad G = \begin{pmatrix} \rho v \\ \rho uv \\ \rho v^2 + p \\ (E+p)v \end{pmatrix} \quad ,$$

are the convective fluxes, and the viscous fluxes are given by

$$F_v = \begin{pmatrix} 0 \\ \tau_{xx} \\ \tau_{xy} \\ u\tau_{xx} + v\tau_{xy} + \frac{\mu}{(\gamma-1)Pr}\frac{\partial a^2}{\partial x} \end{pmatrix} \quad , \quad G_v = \begin{pmatrix} 0 \\ \tau_{xy} \\ \tau_{yy} \\ u\tau_{xy} + v\tau_{yy} + \frac{\mu}{(\gamma-1)Pr}\frac{\partial a^2}{\partial y} \end{pmatrix} .$$

Denoting by $\underline{\tau}$ the viscous stress tensor, the viscous terms in the momentum equations can be written in the form $div\underline{\tau}$. One way of extending the χ-formulation to the Navier-Stokes is to define a monotonic function of the viscous terms as follows

$$\chi_M(div\underline{\tau}) = \alpha_M(\|div\underline{\tau}\|) \cdot div\underline{\tau} \quad , \tag{2.5}$$

where, for a given $\delta > 0$ and $\sigma > 0$,

$$\alpha_M(\|s\|) = \begin{cases} 1 & \text{if } \|s\| \geq \delta \ , \\ f(\|s\|) & \text{if } \delta - \sigma < \|s\| < \delta \ , \\ 0 & \text{if } \|s\| \leq \delta - \sigma \ , \end{cases} \tag{2.6}$$

with $f(\|s\|)$ any smooth monotonic function, with values between 0 and 1.

In the total energy equation, the scalar diffusive term can be written in the form

$$-divq + div(\underline{\tau} \cdot V) = -divq + gradV \cdot \underline{\tau} + V \cdot div\underline{\tau} \quad , \tag{2.7}$$

with q the heat conduction vector and V the velocity vector. Two distinct dissipative mechanisms are present: the heat diffusion $divq$, and the part due to the viscous stress tensor $\underline{\tau}$. The quantity $V \cdot div\underline{\tau}$ is negligible if the viscous terms of the momentum equation $div\underline{\tau}$ are negligible. Therefore, the same function χ_M given in (2.5) can be applied to these terms. In the present application we are interested in subsonic and transonic adiabatic flows along walls. For such flows, the dissipation function $gradV \cdot \underline{\tau}$ is always small with respect to the term

$V \cdot div_{\underline{T}}$. Moreover, in the case of air, the ratio between the thickness of the thermal boundary layer and the thickness of the velocity boundary layer is of order one. In this case, the function χ_M is also a good indicator of the magnitude of the term *divq*. From these considerations we argue that for the applications in which we are interested, the domain decompositon criterion can be based on the χ-function χ_M, and the χ-formulation of problem (2.4) reads

$$\frac{\partial Q}{\partial t} + \frac{\partial F}{\partial x} + \frac{\partial G}{\partial y} = \frac{\alpha_M}{Re}\left(\frac{\partial F_v}{\partial x} + \frac{\partial G_v}{\partial y}\right) \quad . \tag{2.8}$$

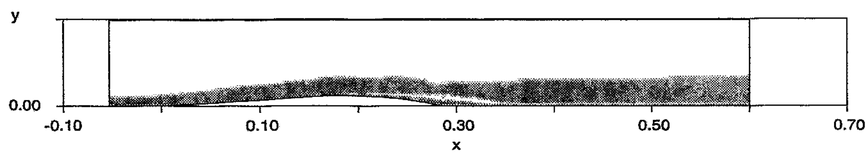

FIGURE 1. Laminar subsonic flow, $Re = 5.10^5$, $\frac{p_{exit}}{p_{tot}} = 0.95$.

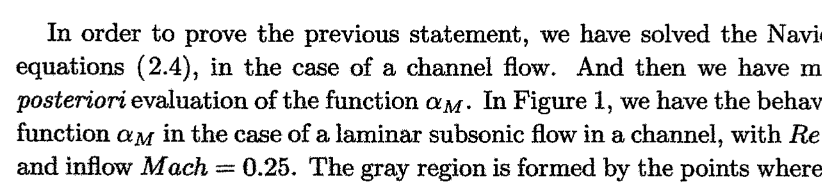

FIGURE 2. Turbulent transonic flow, $Re = 2.3\,10^7$, $\frac{p_{exit}}{p_{tot}} = 0.64$.

In order to prove the previous statement, we have solved the Navier Stokes equations (2.4), in the case of a channel flow. And then we have made an *a posteriori* evaluation of the function α_M. In Figure 1, we have the behavior of the function α_M in the case of a laminar subsonic flow in a channel, with $Re = 5.10^5$, and inflow $Mach = 0.25$. The gray region is formed by the points where $\alpha_M = 1$. Similarly in Figure 2, for the turbulent transonic flow ($Re = 2.3\,10^7$, and inflow $Mach = 0.65$). In both cases, $\delta = 1$ and $\sigma = \delta/100$. From the figures it may be observed that the boundary layer region is well detected, as well as the laminar separation bubble developing behind the bump. In the transonic case, we have a very thin turbulent boundary layer, and a complex shock wave structure, creating downstream a region with vorticity. All these features are detected by the χ-function.

3. Domain Decomposition via the χ Formulation

The previous calculation can be performed by splitting the domain into two parts. A viscous region along the wall, where problem (2.8) is solved, and an inviscid region, formed by the rest of the domain, from the interface to the upper boundary, along the channel symmetry line. In the inviscid region we solve the Euler equations, that is system (2.4) with $F_v = G_v = 0$.

The interface between the two regions is placed in such a way that the resulting viscous region contains the points where the function $\alpha_M \neq 0$, in addition $\alpha_M = 0$ on the grid points belonging to the interface itself. This last requirement along the interface, is very important in order to have a coupling between the Euler equations and the χ-Navier-Stokes equations, of inviscid type (that is Euler-Euler coupling). In this way it is possible to specify boundary conditions consistent with the hyperbolic character of the equations on both sides. By a one-dimensional analysis along the normal to the interface, it is possible to see that for the Euler equations we have to impose the velocity component along the normal and, if there is an incoming flow with respect to the inviscid region, the total enthalpy and the entropy. For the χ-Navier-Stokes equations, we have to specify the pressure and, if the flow is incoming with respect to the viscous subdomain, the total enthalpy and the entropy. We can see that the external inviscid flow is driving the development of the viscous region, by imposing the pressure distribution. And the viscous region interacts with the external inviscid flow by introducing displacement effects. This kind of interaction is well known from the classical boundary layer theory. However, in the present approach we do not introduce any kind of approximation, and the interaction between the two fields is much more general.

The Euler equations, and the χ-Navier-Stokes equations, are solved by a finite-volume method in curvilinear coordinates. The convective terms of the Euler equations are discretized in space by a second order flux vector splitting technique, while for the χ-Navier-Stokes equations a centred scheme is applied. Upwind methods, such as the flux-vector splitting, are very well suited for representing shock waves, but they are excessively dissipative inside the boundary layers. On the contrary, centred schemes give better results inside the viscous layers, but they present problems when capturing shock waves. With the present approach we try to combine the good properties of the two schemes, avoiding their drawbacks. The governing equations are integrated in time by an implicit technique.

After a first calculation of the Navier-Stokes equations, on the whole domain with a coarse grid, we detect the viscous region and we locate the interface following the requirements explained above. Introducing a fine grid in the viscous subdomain, the Euler equations and the χ-Navier-Stokes equations are alternatively integrated in time in each subdomain. After a step in one subdomain, the appropriate characteristic boundary conditions are imposed at the interface, and the other subdomain is calculated. During the coupled calculation, the interface is displaced if the above requirements are not fullfilled, that is if the size of the viscous region was underestimated. Similarly if the viscous region is overestimated, the interface is displaced, in order to optimize the calculation effort.

Complete details on the numerical algorithm, its efficiency, as well as further results, will be presented in a forthcoming paper.

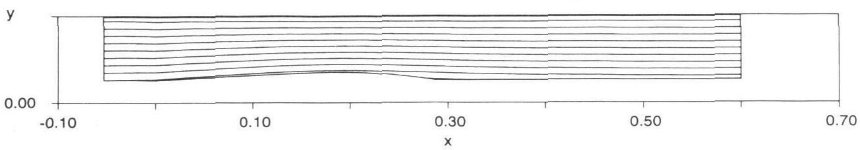

FIGURE 3. Laminar subsonic flow, $Re = 5.10^5$, $\frac{p_{exit}}{p_{tot}} = 0.95$, Inviscid subdomain.

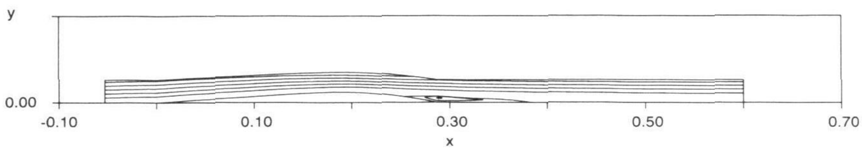

FIGURE 4. Laminar subsonic flow, $Re = 5.10^5$, $\frac{p_{exit}}{p_{tot}} = 0.95$, Viscous subdomain.

In Figure 3 and 4, we report the calculation of the laminar subsonic channel flow. Constant mass-flow lines are reported for the inviscid part of the domain (Figure 3) and the viscous subdomain (Figure 4). The interface was placed along a curvilinear coordinate line, and the mesh size was 11×41 points in the inviscid part, and 31×81 in the viscous subdomain. The accuracy of the solution oobtained by domain decomposition, is comparable with a solution of the Navier-Stokes equations, with a global grid of size 101×81.

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