Domain Decomposition Preconditioners for Elliptic Problems in Two and Three Dimensions: First Approach

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Abstract. In this talk, we shall describe some domain decomposition preconditioners for elliptic boundary value problems in two and three dimensions. We consider the case where more than two subdomains meet at an interior point of the original domain; this allows a subdivision into an arbitrary number of subdomains without the deterioration of the iterative convergence rates of the resulting algorithms. The described preconditioners (for both two and three dimensional applications) result in preconditioned systems whose condition number growth is bounded by $c(1+\ln^2(d/h))$. Here h is the mesh size and d is roughly the size of the largest subdomain. We finally give a technique which utilizes the earlier described methods to derive even more efficient preconditioners. This technique leads to preconditioned systems whose condition number remains bounded independently of the number of unknowns.

1. INTRODUCTION

The need for modeling more complex physical processes has led to the development of larger and faster computers. In the next generation of machines, parallel computing architectures will be employed to gain additional computational improvement. If significant computational improvements are to be realized, then

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algorithms especially tailored to parallel environments must be developed. Moreover, these algorithms should be effective on machines with a large number of processors.

Preconditioners based on domain decomposition gives rise to an important approach for the development of parallel algorithms for elliptic boundary value problems [**1-11**]. Such algorithms are straightforward to efficiently implement on actual parallel computers. In these implementations, a greater number of subdomains gives rise to a greater number of independent parallel tasks leading to an effective use of parallel resources. The overall efficiency of the resulting algorithm also depends upon the rate of iterative convergence which can be estimated in terms of the condition number of the preconditioned system. Accordingly, to be effective in a parallel environment, the conditioning of the preconditioned system should not deteriorate as the number of subregions (i.e. processors) increase. In this talk, I will consider domain decomposition methods whose conditioning improves with the number of subdomains. These methods have been developed jointly with J.H. Bramble and A.H. Schatz of Cornell University.

The first type of domain decomposition preconditioner which we will consider is that described in [**5**]. That paper provided the first example of a domain decomposition strategy whose conditioning improved with the number of subdomains. Loosely, this conditioning improvement is a result of the introduction of a certain 'coarse grid problem', where the coarse grid coincides with the subdomain division of the original domain. That paper also gave an in depth convergence analysis for the method while developing basic analytical techniques for the analysis of domain decomposition algorithms. We will review the algorithm developed in [**5**] and the corresponding analytical results.

We then develop a natural extension of the two dimensional algorithm to three dimensional problems [**8**]. The solution of the resulting preconditioning problem requires the solution of a somewhat complex boundary system. However, this boundary system can be efficiently solved using a technique described in [**6,8**]. We have shown that the corresponding preconditioned systems have condition number growth bounded by $C(1 + \ln^2(d/h))$ where h is the mesh size and d is roughly the size of the subdomains. Thus, the conditioning improves as more subdomains are used (i.e. d becomes smaller).

We finally consider technique which can improve the computational efficiency of the previously described preconditioning methods. A new class of preconditioners was defined in [**7**] which led to preconditioned systems with bounded conditioning. This class of preconditioners utilized the previously defined domain decomposition algorithms and a computationally lower order boundary modification. We shall describe this boundary modification.

Another approach to the development of domain decomposition preconditioners was given in [**8,11**]. This approach gives rise to a domain decomposition strategies for two and three dimensional problems whose conditioning also improves with the number of subdomains. However, in this case, the improvement is a result of the introduction of an 'average value problem' where each subdomain has a corresponding 'average value.' We will not consider this approach in any further detail here.

The outline of the remainder of the paper is as follows. In Section 2, we describe the model elliptic problem and formulate the preconditioning problem in terms of forms. Section 3 describes the two dimension domain decomposition technique of [5]. Its natural extension to three dimensional problems is given in Section 4. In Section 5, we describe a technique for improving the computational efficiency of the method of Section 3. Finally, in Section 6, we discuss implementation aspects of the earlier described preconditioners.

2. PRELIMINARIES

In this section, we shall describe the elliptic problem and corresponding Hilbert spaces in which it is posed. We also describe the preconditioning problem in terms of the definition of an appropriate form.

We shall restrict ourselves to boundary value problems in R^N for $N = 2$ and $N = 3$. Let Ω be a bounded domain in R^N with a piecewise smooth boundary $\partial\Omega$. As a model problem for a second order uniformly elliptic equation we shall consider the Dirichlet problem

$$\begin{aligned} Lu &= f && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{2.1}$$

where

$$Lv = -\sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}(a_{ij}\frac{\partial v}{\partial x_j}),$$

with a_{ij} uniformly positive definite, bounded and piecewise smooth on Ω. The generalized Dirichlet form is given by

$$A(v,\phi) = \sum_{i,j=1}^{N} \int_\Omega a_{ij} \frac{\partial v}{\partial x_i}\frac{\partial \phi}{\partial x_j}\, dx, \tag{2.2}$$

which is defined for all v and ϕ in the Sobolev space $H^1(\Omega)$ (the space of distributions with square integrable first derivatives). The $L^2(\Omega)$ inner product is denoted

$$(v,\phi) = \int_\Omega v\phi\, dx.$$

The subspace $H_0^1(\Omega)$ is the completion of the smooth functions with support in Ω with respect to the norm in $H^1(\Omega)$. The weak formulation of the problem defined by (2.1) is: Find $u \in H_0^1(\Omega)$ such that

$$A(u,\phi) = (f,\phi) \tag{2.3}$$

for all $\phi \in H_0^1(\Omega)$. This leads immediately to the standard Galerkin approximation. Let $S_h^0(\Omega)$ be a finite dimensional subspace of $H_0^1(\Omega)$. The Galerkin approximation is defined as the solution of the following problem: Find $U \in S_h^0(\Omega)$ such that

$$A(U,\Phi) = (f,\Phi) \tag{2.4}$$

for all $\Phi \in S_h^0(\Omega)$.

We shall be concerned in this talk with the efficient solution of (2.4) using preconditioners based on domain decomposition. The question of defining a preconditioner for (2.4) can be approached from two points of view. The first requires the choice of a basis for $S_h^0(\Omega)$. Employing this basis, one is led to a matrix problem for the computation of the corresponding coefficients of U. The preconditioning problem from this point of view is to define another matrix which is easier to invert and 'spectrally close' to the original. Alternatively (cf. [**4**,5]), the preconditioning problem can be viewed as a problem of defining a symmetric positive definite quadratic form $B(\cdot,\cdot)$ which is equivalent to $A(\cdot,\cdot)$ on $S_h^0(\Omega) \times S_h^0(\Omega)$. At each step of the iteration we must solve problems of the form: Given a linear functional G on $S_h^0(\Omega)$, find $W \in S_h^0(\Omega)$ such that

$$B(W,\omega) = G(\omega) \qquad \text{for all } \omega \in S_h^0(\Omega). \tag{2.5}$$

The problem of finding the solution of (2.5) should be computationally less complex than that of finding the solution to (2.4) on the given computer architecture. The corresponding spectral condition in terms of forms reduces to inequalities of the form

$$c_0 B(\omega,\omega) \le A(\omega,\omega) \le c_1 B(\omega,\omega) \qquad \text{for all } \omega \in S_h^0(\Omega). \tag{2.6}$$

The condition number of the preconditioned system is bounded by c_1/c_0. In the above inequalities, c_0 and c_1 are constants which may depend on d (the subdomain size) and h.

3. A Two Dimensional Example

In this section, we describe the two dimensional domain decomposition preconditioner given in [**5**]. We shall give a simplified description of the algorithm under certain mesh assumptions. Applications to more general situations are given in [**5**].

We shall make the following assumptions with regard to Ω (cf. [**5**] for details). First, assume that Ω is a polygonal domain and that, for each h, $0 < h < 1$ a parameter, Ω has been given a quasi-uniform triangulation Ω^h. We assume that $\Omega = \cup\Omega_k$ may be written as the union of n_r disjoint quadrilaterals Ω_k of quasi-uniform size $d \ge h$. The boundary of the subdomains should be part of the mesh boundary Ω^h. This means that any given mesh triangle in $\{\tau_j\}$ is contained in some $\bar{\Omega}_j$. The collection of regions $\{\Omega_k\}$ will frequently be referred to as the subdomains.

The vertices of the $\{\Omega_k\}$ will be labeled v_j (ordered in some way) and Γ_{ij} will denote the straight line segment with endpoints v_i and v_j. Throughout this paper we shall only consider Γ_{ij} when Γ_{ij} is an edge of some Ω_k. Furthermore, we associate with each Ω_k the triangulation inherited from the original triangulation Ω^h. The example given in Figure 3.1 should help clarify the situation.

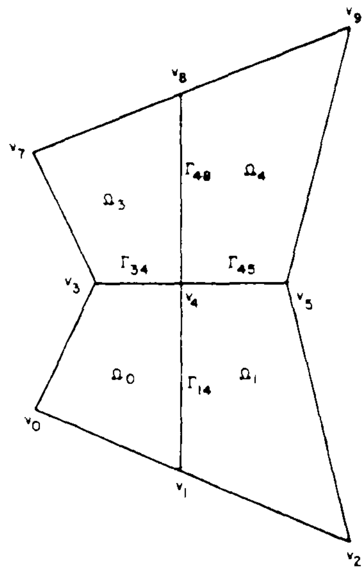

Figure 3.1. The domain Ω and subdomains.

For the purposes of this talk, we shall make the additional assumptions that:
(1) The mesh on any edge Γ_{ij} is uniformly spaced.
(2) We are willing to solve Dirichlet subproblems corresponding to the original form $A(\cdot,\cdot)$ on the subdomains (see (6.1)).

For each h, let $S_h(\Omega)$ be the space of continuous piecewise linear functions defined relative to the triangulation Ω^h and $S_h^0(\Omega)$ be the subspace of $S_h(\Omega)$ consisting of those functions which vanish on $\partial\Omega$. $S_h^0(\Omega_j)$ will denote the subspace of $S_h^0(\Omega)$ of functions whose supports are contained in $\bar{\Omega}_j$ (in particular, they vanish on $\partial\Omega_j$ and outside $\bar{\Omega}_j$). In addition, $S_h(\Omega_j)$ will be the set of functions which are restrictions of those in $S_h^0(\Omega)$ to $\bar{\Omega}_j$. Subspaces on the boundaries of the subdomains will be denoted as follows. $S_h(\partial\Omega_j)$ will denote the restrictions of $S_h(\Omega_j)$ to $\partial\Omega_j$ and $S_h^0(\Gamma_{ij})$ will be the subspace of $S_h(\partial\Omega_j)$ consisting of functions whose support is contained on the edge Γ_{ij}. Finally set $\Gamma = \cup\partial\Omega_i$.

In what follows, c and C (with or without subscript) will denote generic positive constants which are independent of h, d and the Ω_k.

We construct our preconditioner B by constructing its corresponding bilinear form $B(\cdot,\cdot)$ defined on $S_h^0(\Omega) \times S_h^0(\Omega)$. We first decompose functions in $S_h^0(\Omega)$ as follows: Write $W = W_P + W_H$ where $W_P \in S_h^0(\Omega_1) \oplus \ldots \oplus S_h^0(\Omega_{n_r})$ and satisfies

$$A(W_P, \Phi) = A(W, \Phi) \qquad \text{for all } \Phi \in S_h^0(\Omega_k)$$

for each k. Notice that W_P is determined on Ω_k by the values of W on Ω_k and that

$$A(W_H, \Phi) = 0, \qquad \text{for all } \Phi \in S_h^0(\Omega_k).$$

Thus on each Ω_k, W is decomposed into a function W_P which vanishes on $\partial\Omega_k$ and a function W_H which satisfies the above homogeneous equations and has the same boundary values as W. We shall refer to such a function W_H as "discrete A-harmonic ".

We note that the above decomposition is orthogonal in the A-inner product and hence

$$A(W,W) = A(W_P,W_P) + A(W_H,W_H). \tag{3.1}$$

We shall define $B(\cdot,\cdot)$ by replacing the $A(W_H,W_H)$ term above. To do this, we decompose W_H into $W_H = W_E + W_V$, where $W_V \in S_h^0(\Omega)$ is the discrete A-harmonic function which is linear on each edge Γ_{ij} and has the same values as W at the vertices. Thus W_E is a discrete A-harmonic function in Ω_k for each k which vanishes at all of the vertices.

To define the form $B(\cdot,\cdot)$, we need to define a discrete operator $l_0^{1/2}$. Let Γ_{ij} be an edge of Γ. We first define $l_0 : S_h^0(\Gamma_{ij}) \mapsto S_h^0(\Gamma_{ij})$ by $l_0\theta = \eta$ where η solves

$$\int_{\Gamma_{ij}} \eta\omega\, dx = \int_{\Gamma_{ij}} \theta'\omega'\, dx \qquad \text{for all } \omega \in S_h^0(\Gamma_{ij}). \tag{3.2}$$

The operator l_0 is symmetric positive definite on $S_h^0(\Gamma_{ij})$ and $l_0^{1/2}$ is defined to be its square root.

We define the form $B(\cdot,\cdot)$ by

$$\begin{aligned} B(W,\Phi) &= A(W_P,\Phi_P) + \sum_{\Gamma_{ij}} < l_0^{1/2} W_E, \Phi_E >_{\Gamma_{ij}} \\ &\quad + \sum_{\Gamma_{ij}} (W_V(v_i) - W_V(v_j))(\Phi_V(v_i) - \Phi_V(v_j)). \end{aligned} \tag{3.3}$$

In (3.3), $\langle\cdot,\cdot\rangle_{\Gamma_{ij}}$ denotes the L^2 inner product on Γ_{ij}. The following theorem is proved in [**5**]:

THEOREM 1. *Let B be defined by (3.3). There are positive constants λ_0, λ_1 and C such that*

$$\lambda_0 B(W,W) \le A(W,W) \le \lambda_1 B(W,W) \qquad \textit{for all } W \in S_h^0(\Omega),$$

where

$$\frac{\lambda_1}{\lambda_0} \le C\ (1 + \ln^2(d/h)).$$

If all of the vertices of the Ω_k lie on Γ then

$$\frac{\lambda_1}{\lambda_0} \le C.$$

4. THE THREE DIMENSIONAL FORMULATION

In this section, we describe a natural extension of the domain decomposition preconditioner of Section 3 to three dimensions. For simplicity of presentation, we shall consider applications to the unit cube. We subdivide Ω into an equally spaced mesh of rectangular prisms $\Omega = \cup\tau_j$ and define $S_h(\Omega)$ to be the set of functions which are continuous on Ω and piecewise tri-linear on the τ_j.

To define domain decomposition algorithms, we must start with a decomposition of Ω into n_r subdomains, $\Omega = \cup_{i=1}^{n_r}\Omega_i$. For simplicity, subdomains can be thought of as rectangular prisms. The subregions are to be 'quasi-uniform' of size d with boundaries that align with the mesh $\cup\tau_j$.

In this section, Γ_{ij} will denote the face between subregions i and j. Except for this one change, we shall use the same notation as that of Section 3. In addition, we define

$$\delta_i = \cup\partial\Gamma_{ij}$$

where the union is over faces Γ_{ij} of $\partial\Omega_i$.

To define the domain decomposition preconditioner, we will again replace the $A(W_H, W_H)$ term in (3.1). To do this, we decompose $W_H \in S_h^0(\Omega)$ into $W_H = W_F + W_E$, where W_E is the function in $S_h^0(\Omega)$ which

(1) equals W_H on $\delta \equiv \cup\delta_i$,

(2) is 2-dimension discrete harmonic on the faces of the subdomains,

(3) and is discrete A-harmonic in Ω_k for each k.

By (2), we mean that on each face Γ_{ij}, W_E satisfies the homogeneous equations

$$\int_{\Gamma_{ij}} \nabla W_E \cdot \nabla\phi \, dx = 0$$

for all $\phi \in S_h^0(\Gamma_{ij})$. Note that W_F is a discrete A-harmonic function in Ω_k which vanishes on δ_k for each k.

We next define the corresponding $l_0^{1/2} : S_h^0(\Gamma_{ij}) \mapsto S_h^0(\Gamma_{ij})$ operator. This is completely analogous to the two

dimensional definition except that (3.2) gets replaced by

$$\int_{\Gamma_{ij}} \eta\omega \, dx = \int_{\Gamma_{ij}} \nabla\theta \cdot \nabla\omega \, dx \qquad \text{for all } \omega \in S_h^0(\Gamma_{ij}).$$

This operator will be used to define the form on the 'face' function W_F.

Let $\bar{W}_E^i$ denote the average value of W_E over the nodes of $\cup\tau_j$ on δ_i. The preconditioning form $B(\cdot,\cdot)$ is defined by

$$B(W,W) = A(W_P, W_P) + \sum_{\Gamma_{ij}} < l_0^{1/2} W_F, W_F >_{\Gamma_{ij}} + \sum_i \left|W_E - \bar{W}_E^i\right|_{d,\delta_i}^2 . \tag{4.1}$$

Here $|\cdot|_{d,\delta_i}^2$ denotes the discrete norm

$$|v|_{d,\delta_i}^2 = h \sum_j v(x_j)^2. \tag{4.2}$$

The sum in (4.2) is taken over the nodes x_j on δ_i. The following theorem is proved in [8].

THEOREM 2. *Let B be defined by (4.1). There are positive constants c_0 and c_1, not depending on d or h, such that*

$$c_0(1+\ln^2(d/h))^{-1}B(W,W) \le A(W,W) \le c_1 B(W,W) \qquad \text{for all } W \in S_h^0(\Omega).$$

5. THE IMPROVED METHOD

In this section, we describe a technique which utilizes the previously described preconditioners to develop an even more efficient preconditioner [**7**]. Loosely, the conditioning properties of the form B are improved by replacing the boundary form by an appropriate preconditioned iteration. We shall only discuss the application of this technique in the two dimensional case.

Fix a subdomain Ω_i. We first need to define a 'loop' operator $l^{1/2}$. Define $l : S_h(\partial\Omega_i) \mapsto S_h(\partial\Omega_i)$ by $l\theta = \eta$ where η solves

$$\int_{\partial\Omega_i} \eta\omega\, dx = \int_{\partial\Omega_i} \theta'\omega'\, dx \qquad \text{for all } \omega \in S_h(\partial\Omega_i),$$

where the primes denote differentiation with respect to arc length along each side of $\partial\Omega_i$. The operator l is symmetric and non-negative on $S_h(\partial\Omega_i)$ and we define $l^{1/2}$ to be its square root.

It is shown in [**7**], that

$$c\langle QW_H, W_H\rangle_\Gamma \le A(W_H, W_H) \le C\langle QW_H, W_H\rangle_\Gamma,$$

holds for discrete A-harmonic functions W_H where

$$\langle QW_H, W_H\rangle_\Gamma \equiv \sum_i \left\langle l^{1/2}W_H, W_H\right\rangle_{\partial\Omega_i}.$$

Accordingly, $\langle QW_H, W_H\rangle_\Gamma$ is a good candidate for a replacement for $A(W_H, W_H)$. Unfortunately, problems involving the $\langle Q\cdot,\cdot\rangle_\Gamma$ form cannot be easily solved hence this replacement does not lead to an effective algorithm. We note, however, that the action of the form $\langle Q\cdot,\cdot\rangle_\Gamma$ can be efficiently evaluated (cf. [**7**]).

Instead, we replace $A(W_H, W_H)$ as follows. Let $\tilde{Q}$ be another positive definite symmetric operator on $S_h^0(\Gamma)$ (the restrictions to Γ of functions in $S_h^0(\Omega)$). Define the operator $\bar{Q}^{-1}$ by

$$\bar{Q}^{-1} = P_m(\tilde{Q}^{-1}Q)\tilde{Q}^{-1},$$

where P_m is an appropriately chosen polynomial of degree m. This polynomial is related to the classical Chebyschev polynomials and is defined in [**7**] so that $\bar{Q}^{-1}$ is positive definite and $\bar{Q}$ is uniformly equivalent to Q on $S_h^0(\Gamma)$, i.e.

$$c_0\left\langle \bar{Q}V, V\right\rangle_\Gamma \le \langle QV, V\rangle_\Gamma \le c_1\left\langle \bar{Q}V, V\right\rangle_\Gamma \quad \text{for all} \quad V \in S_h^0(\Gamma). \tag{5.1}$$

Hence, we define B by

$$B(W,W) = A(W_P, W_P) + \left\langle \bar{Q}W_H, W_H\right\rangle_\Gamma. \tag{5.2}$$

By (5.1) and (3.1)

$$C_0B(W,W) \le A(W,W) \le C_1B(W,W) \qquad \text{for all } W \in S_h^0(\Omega). \tag{5.3}$$

As described above, the operator $\tilde{Q}$ is arbitrary. An effective choice of $\tilde{Q}$ results from considering the boundary part of the form given in Section 3. That is, we define $\tilde{Q}$ by $\tilde{Q}\theta = \eta$ where η solves

$$\langle \eta, \omega \rangle_\Gamma = \sum_{\Gamma_{ij}} < l_0^{1/2}\theta_E, \omega_E >_{\Gamma_{ij}} + \sum_{\Gamma_{ij}} (\theta(v_i) - \theta(v_j))(\omega(v_i) - \omega(v_j)). \tag{5.4}$$

We shall discuss algorithms for solving (2.5) with B given by (5.3)-(5.4) in Section 6. Let us however note that degree of the polynomial needed to obtain (5.1) is proportional to the square root of the condition number K of $\tilde{Q}^{-1}Q$. When defining $\tilde{Q}$ by (5.4), Theorem 1 gives the bound

$$K \le C(1 + \ln^2(d/h)).$$

In the computational experiments given in [**7**], m equal to two or three usually sufficed.

The improvement in efficiency using the preconditioner defined by (5.3)-(5.4) is the result of two properties. First, as noted above, the degree of P_m need not be large. Secondly, the work per term in evaluating P_m is not large. Indeed, under appropriate assumptions, the work is proportional (up to a logarithm) to the number of nodes on Γ (cf. [**7**])

6. THE SOLUTION OF (2.5)

We use a three step algorithm to compute the solution $W = W_P + W_H$ of (2.5) (cf. [**4,5**]). The function W_P restricted to Ω_k is a function in $S_h^0(\Omega_k)$ and satisfies

$$A(W_P, \Phi) = G(\Phi) \qquad \text{for all } \Phi \in S_h^0(\Omega_k). \tag{6.1}$$

Thus the function W_P on Ω_k can be obtained by solving the corresponding Dirichlet problem on Ω_k (6.1). Note that the problems on different subdomains are independent of each other and hence can be solved in parallel.

Now with W_P known, we are left with the problem of finding W_H. This is accomplished in two steps. First, the values of W_H are computed on Γ. This step will be considered in more detail later. Once the boundary values of W_H are known on Γ, we need only compute the values of the discrete A-harmonic on the interior of the subdomains. As described in [**4,5**], this problem can be reduced to the solution of independent Dirichlet problems on the subdomains which can be solved in parallel.

We next consider the second step, i.e. given W_P find the values of W_H on Γ. Let Φ be an arbitrary function in $S_h^0(\Gamma)$. For the preconditioner given by (3.3), we must solve

$$\sum_{\Gamma_{ij}} < l_0^{1/2} W_E, \Phi_E >_{\Gamma_{ij}} + \sum_{\Gamma_{ij}} (W_V(v_i) - W_V(v_j))(\Phi_V(v_i) - \Phi_V(v_j)) \\ = G(\tilde{\Phi}) - A(W_P, \tilde{\Phi}), \tag{6.2}$$

where $\bar{\Phi}$ is any extension of Φ in $S_h^0(\Omega)$. By an appropriate choice of basis functions (cf. [5]), W_H on Γ can be computed by inverting the operator $l_0^{1/2}$ on each edge and solving a coarse difference problem for the corner values of W_V. All of these processes are independent and can be done in parallel.

For the preconditioner given by (4.1), to compute W_H, we must solve

$$(6.3)\qquad \sum_{\Gamma_{ij}} < l_0^{1/2} W_F, \Phi_F >_{\Gamma_{ij}} + \sum_i \langle W_E - \bar{W}_E^i, \Phi_E \rangle_{d,\delta_i} = G(\bar{\Phi}) - A(W_P, \bar{\Phi}).$$

In (6.3), $\langle \cdot, \cdot \rangle_{d,\delta_i}$ denotes the discrete inner product corresponding to (4.2). Again, by an appropriate choice of basis function, one can compute W_F by inverting $l_0^{1/2}$ on each face. The calculations on the individual faces can be done in parallel. It is also possible to derive a symmetric positive definite sparse system of linear equations for the computation of $\bar{W}_E^i$ (cf. [6,8]). Thus to compute W_E, we first solve this $n_r \times n_r$ system for the values of $\bar{W}_E^i$. When these values are known, the nodal values of W_E on Γ can then be computed by applying the inverse of a diagonal matrix [8,11].

We finally consider the problem of computing the values of W_H when B is given by (5.2). By the definition of $\bar{Q}$, W_H on Γ is given by

$$(6.4)\qquad W_H = P_m(\tilde{Q}^{-1}Q)\tilde{Q}^{-1}V_H.$$

where V_H solves

$$(6.5)\qquad \langle V_H, \theta \rangle_\Gamma = G(\bar{\theta}) - A(W_P, \bar{\theta}) \qquad \text{for all } \theta \in S_h^0(\Gamma)$$

and $\bar{\theta}$ is any extension of θ in $S_h^0(\Omega)$. Note that $\tilde{Q}^{-1}V_H$ is the solution to

$$\left\langle \tilde{Q} V_H, \theta \right\rangle_\Gamma = G(\bar{\theta}) - A(W_P, \bar{\theta}) \qquad \text{for all } \theta \in S_h^0(\Gamma),$$

and given $\varsigma \in S_h^0(\Gamma)$, $\eta = \tilde{Q}^{-1}Q\varsigma$ solves

$$(6.6)\qquad \left\langle \tilde{Q}\eta, \theta \right\rangle_\Gamma = \langle Q\varsigma, \theta \rangle_\Gamma \qquad \text{for all } \theta \in S_h^0(\Gamma).$$

Accordingly, the computation of W_H on Γ only requires evaluation of the form $\langle Q\varsigma, \cdot \rangle_\Gamma$ and the solution of problems of the form

$$\left\langle \tilde{Q}\eta, \theta \right\rangle_\Gamma = F(\theta) \qquad \text{for all } \theta \in S_h^0(\Gamma),$$

where $F(\theta)$ is a known linear functional. The evaluation of the right hand side of (6.6) is discussed in [7]. However, we note that the cost of each evaluation is proportional to a logarithm of d/h times the number of nodes on Γ. The inversion of $\tilde{Q}$ is similar to the computation of W_E and W_V on Γ solving (6.2).

REFERENCES

1 P.E. Bjørstad and O.B. Widlund, *Solving elliptic problems on regions partitioned into substructures*, "Elliptic Problem Solvers II," (eds, G. Birkhoff and A. Schoenstadt), Academic Press, New York, 1984, pp. 245–256.

2 P.E. Bjørstad and O.B. Widlund, *Iterative methods for the solution of elliptic problems on regions partitioned into substructures*, SIAM J. Numer. Anal. **23** (1986), 1097–1120.

3 J.H. Bramble, R.E. Ewing, J.E. Pasciak and A.H. Schatz, *A preconditioning technique for the efficient solution of problems with local grid refinement*, Comp. Meth. Appl. Mech. Eng. (to appear).

4 J.H. Bramble, J.E. Pasciak and A.H. Schatz, *An iterative method for Elliptic problems on regions partitioned into substructures*, Math. Comp. **46** (1986), 361–369.

5 J.H. Bramble, J.E. Pasciak and A.H. Schatz, *The construction of preconditioners for elliptic problems by substructuring, I*, Math. Comp. **47** (1986), 103–134.

6 J.H. Bramble, J.E. Pasciak and A.H. Schatz, *The construction of preconditioners for elliptic problems by substructuring, II*, Math. Comp. (to appear).

7 J.H. Bramble, J.E. Pasciak and A.H. Schatz, *The construction of preconditioners for elliptic problems by substructuring, III*, (preprint).

8 J.H. Bramble, J.E. Pasciak and A.H. Schatz, *The construction of preconditioners for elliptic problems by substructuring, IV*, (in preparation).

9 Q.V. Dihn, R. Glowinski and J. Périaux, *Solving elliptic problems by domain decomposition methods*, in "Elliptic Problem Solvers II," (eds, G. Birkhoff and A. Schoenstadt), Academic Press ', New York, 1984, pp. 395–426.

10 G.H. Golub and D. Meyers, *The use of preconditioning over irregular regions*, "Proceedings of the Sixth International Conference on Computing Methods in Science and Engineering" (to appear).

11 J.E. Pasciak, *Domain decomposition preconditioners for elliptic problems in two and three dimensions*, in "Proceedings, Numerical Algorithms for Modern Parallel Computer Architectures," Inst. Math. and Applic., Univ. of Minnesota, (to appear).