A Method of Domain Decomposition for Three-Dimensional Finite Element Elliptic Problems

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Abstract. We describe new results on domain decomposition methods for second-order 3-D elliptic problems which recently has been obtained jointly with Olof Widlund. The original region Ω of the elliptic problem is partitioned into a few or a large number of subregions Ω_i by cuts which intersect each other. In our iterative algorithms the Dirichlet and mixed subproblems with Dirichlet data on the edges of Ω_i are solved. In the case of a large number of subregions the Dirichlet boundary conditions on the edges are replaced by discrete average values. An alternative algorithm is based on the piecewise linear functions defined on the coarse triangulation. The subproblems in these algorithms are independent and may be solved in parallel.

1. Introduction. In this paper, we will discuss results on domain decomposition methods for solving linear systems of algebraic equations resulting from the finite element approximation to self-adjoint, second-order 3-D elliptic problems which recently has been obtained jointly with Olof Widlund. We will discuss the case when the original region Ω of the elliptic problem is partitioned into a few or a large number of subregions Ω_i.

In our iterative algorithms we will solve subproblems which are the Dirichlet and mixed problems on subregions

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Ω_i . The boundary conditions for the mixed problems are of Neumann type except on the edges, the so called wire basket, where Dirichlet conditions are imposed. These subproblems are independent and may be solved in parallel. In the case of a large number of subregions the Dirichlet boundary conditions on the wire basket should be modified by discrete average values. An alternative algorithm is based on the piecewise linear functions defined on the coarse triangulation. These methods lead to preconditioned systems with condition numbers proportional to $(1+\log\frac{H}{h})^3$ and $\frac{H}{h}(1+\log\frac{H}{h})$ respectively, where H and h are parameters associated with the coarse and fine triangulations.

Our results are generalizations of results from [5], [6] in which 2-D elliptic problems are discussed.

A domain decomposition method for 3-D elliptic problems has recently been presented in [3] . That method is also based on the discrete average values on $\partial\Omega_i$ and leads to a preconditioned system with a condition number proportional to H/h. We also note that domain decomposition methods for 2-D elliptic problems have been discussed in many papers. For references to the literature see [1] and [11] .

Throughout this paper C or C_i will denote a positive, generic constant independent of H and h .

2. Statement of problems. We consider the following weak form of the Dirichlet problem for second order elliptic equations.

For $f \in L^2(\Omega)$ find a function $u \in H_0^1(\Omega)$ such that

$$a(u,v) = l(v)\ , \qquad v \in H_0^1(\Omega)\ , \tag{2.1}$$

where Ω is a bounded region in R^3 , and

$$a(u,v) = \int_\Omega \Big(\sum_{i,j=1}^{3} a_{ij}(x) D_i u D_j v + c(x) uv \Big) dx$$

and

$$l(v) = \int_\Omega fv\, dx \quad .$$

We assume that the bilinear form $a(u,v)$ is symmetric, H_0^1-elliptic and continuous.

The problem (2.1) is solved by a finite element method with tetrahedral linear elements. To simplify the presentation, we suppose that Ω is a polyhedral region. Let Ω_h denote a triangulation of Ω with elements e_i and a parameter h . For a given triangulation Ω_h we define the finite element space $V_h^{(1)}$ of piecewise linear, continuous functions and vanishing on $\partial\Omega$.

Find a function $u_h \in V_h^{(1)}(\Omega)$ such that

(2.2) $a(u_h,v_h) = l(v_h)$, $v_h \in V_h^{(1)}$.

Using a natural basis of the space $V_h^{(1)}(\Omega)$ we rewrite the problem (2.2) as the linear system

(2.3) $Au_h = f_h$,

where $(Au_h,v_h)_{R^n} = a(u_h,v_h)$, $(f_h,v_h)_{R^n} = l(v_h)$ and n is the total number of nodal parameters in Ω . The matrix A is positive definite and symmetric.

From now on we will identify $v_h \in V_h^{(1)}(\Omega)$ with its vector representation. We will also drop the subscript h for functions in $V_h^{(1)}(\Omega)$.

3. Domain decomposition method. We will describe and analyze iterative methods for solving the system (2.3) when the region Ω is partitioned into subregions by cuts which intersect each other. We form an auxiliary triangulation Ω_H of Ω with parameter H ($H > h$). It consists of tetrahedrons or cubes (macro-elements, substructures) denoted by Ω_i . Each of them is the union of tetrahedrons e_i of the original triangulation Ω_h . We assume that the triangulations Ω_H and Ω_h of the region Ω are regular (see for example [4] , p. 132).

The subregions Ω_i of Ω are ordered into Neumann - (N) and Dirichlet - (D) type regions as follows in what essentially is a red-black ordering. If two subregions share a face, they must thus be of different types. Of course such ordering is available only for some partitioning but it can easily be accomplished. Let Ω_N and Ω_D denote the union of N-type and D-type subregions. The faces and edges of Ω_i are denoted by F_{ij} and E_{ik} . Let $F = \partial\Omega_N \setminus \partial\Omega$ and $\bar{\Omega}_N = \Omega_N \cup F$.

We represent the matrix A as a 3×3 block matrix of the form

$$(3.1)\quad Au = \begin{bmatrix} A_{11} & 0 & A_{13} \\ 0 & A_{22} & A_{23} \\ A_{13}^T & A_{23}^T & A_{33} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}$$

The matrices A_{11} , A_{22} and A_{33} represent the couplings between pairs of degrees of freedom in Ω_D , Ω_N and F respectively. A_{13} and A_{23} couplings between pairs belonging to Ω_D and F, and Ω_N and F respectively. It is easy to see that

(3.2) $$a(u,v) = a_D(u,v) + a_N(u,v)$$

where a_D and a_N are the restrictions of the bilinear form to Ω_D and Ω_N. The stiffness matrices corresponding to these bilinear form take the form

(3.3) $$A^{(D)} = \begin{bmatrix} A_{11} & A_{13} \\ A_{13}^T & A_{33}^{(1)} \end{bmatrix} \quad \text{and} \quad A^{(N)} = \begin{bmatrix} A_{22} & A_{23} \\ A_{23}^T & A_{33}^{(2)} \end{bmatrix}$$

Our aim is to solve (3.1) using a method of domain decomposition with the exact solvers for the subproblems. Some of our methods will have condition numbers which grow very slowly (logarithmically) with H/h. In our proofs we need a number of lemmas.

4. Auxiliary lemmas. We introduce the Sobolev space $H^{1/2}(0,H)$ of functions defined on $(0,H)$, see [7], [9]. Its norm with weight $1/H$ is of the form

(4.1) $$\|v\|_{H^{1/2}(0,H)} = \left(\int_0^H\int_0^H \frac{|v(x)-v(y)|^2}{|x-y|^2}\,dxdy + \frac{1}{H}\int_0^H v^2(x)dx\right)^{1/2}$$

We note that the weight factor is natural and that it is obtained by transforming from an interval of length 1. Let $(0,H)$ be partitioned into subintervals of length h and let $V_h^{(1)}(0,H)$ denote a finite element space of piece wise linear, continuous functions.

LEMMA 1. For any $v \in V_h^{(1)}(0,H)$,

(4.2) $$\|v\|_{L^\infty(0,H)} \leq C(1+\log\tfrac{H}{h})^{1/2}\,\|v\|_{H^{1/2}(0,H)} .$$

Proof. It suffices to prove (4.2) under the assumption that $H = 1$. Let G be a triangle $0 < x_1 < 1$, $0 < x_2 < 1-x_1$. We construct a triangulation on G with mesh size $\tilde{h} = (H/h)^{-1}$. The function $v \in V_h^{(1)}(0,1)$ is given on $[0,1]$. In view of Lemma 5.6 from [9] there exists an extension such that

$$\|\tilde{v}\|_{H^1(G)} \leq C \|v\|_{H^{1/2}(0,1)}$$

The function $\tilde{v}(x)$ is in general not a piecewise continuous function. Using the extension theorem for finite element functions recently proved in [10] we can construct a piecewise linear, continuous function v_h such that $v_h = \tilde{v} = v$ for $x_1 \in [0,1]$ and

$$\|v_h\|_{H^1(G)} \leq C \|\tilde{v}\|_{H^1(G)}$$

On the other hand, see [2] ,

$$\|v_h\|_{L^\infty(G)} \leq C(1+\log\tfrac{H}{h})^{1/2} \|v_h\|_{H^1(G)}$$

Combining these inequalities, we obtain

$$\|v_h\|_{L^\infty(G)} \leq C(1+\log\tfrac{H}{h})^{1/2} \|v\|_{H^{1/2}(0,1)}$$

which proves (4.2).

COROLLARY of LEMMA 1. If there exists at least one point $Q \in [0,H]$ at which $v(Q) = 0$ then

(4.3) $$\|v\|_{L^\infty(0,H)} \leq C(1+\log\tfrac{H}{h})^{1/2} \|v+\alpha\|_{H^{1/2}(0,H)}$$

for any constant α .

Proof. We have

$$\|v\|_{L^\infty(0,H)} \leq \|v+\alpha\|_{L^\infty(0,H)} + \|\alpha\|_{L^\infty(0,H)} \leq 2\|v+\alpha\|_{L^\infty(0,H)}$$

Applying (4.2) we get (4.3).

We now introduce the Sobolev space $H^{1/2}(F_{ij})$ of functions defined on the faces F_{ij} of Ω_i . By the definition, see [7] , [9] ,

(4.4) $$\|v\|_{H^{1/2}(F_{ij})} = \Big(\int_{F_{ij}}\int_{F_{ij}} \frac{|v(x)-v(y)|^2}{|x-y|^3}\,dx\,dy + \frac{1}{H}\int_{F_{ij}} (v(x))^2 dx\Big)^{1/2}$$

where H is the parameter of the coarse triangulation of

Ω. If v is a smooth function on F_{ij} with support contained in F_{ij}, then (4.4) is equivalent to

$$(4.5)\quad \|v\|_{H_{00}^{1/2}(F_{ij})} = \Big(\int_{F_{ij}} \int_{F_{ij}} \frac{|v(x)-v(y)|^2}{|x-y|^3}\,dxdy + \int_{F_{ij}} \frac{(v(x))^2}{d(x,\partial F_{ij})}\,dx\Big)^{1/2}$$

see [7], [9] where $d(x,\partial F_{ij})$ denotes the distance from x to the boundary ∂F_{ij} of F_{ij}. The completion of the smooth functions with support in F_{ij} with respect to the norm (4.5) is denoted by $H_{00}^{1/2}(F_{ij})$. It can also be defined by interpolating $H_0^1(F_{ij})$ and $L_2(F_{ij})$, i.e.

$$H_{00}^{1/2}(F_{ij}) = \left[H_0^{1/2}(F_{ij}), L_2(F_{ij})\right]_{1/2}$$

see [8].

In the case when F_{ij} is a rectangle $F=(0,H_1)x(0,H_2)$ the norms (4.4) and (4.5) are equivalent to the following, see [9]. Let

$$|v|^2_{H^{1/2}(F)} = \int_0^{H_1}\int_0^{H_1} \frac{\|v(x_1,\cdot)-v(y_1,\cdot)\|^2_{L^2(0,H_2)}}{|x_1-y_1|^2}\,dx_1dy_1 + \int_0^{H_2}\int_0^{H_2} \frac{\|v(\cdot,x_2)-v(\cdot,y_2)\|^2_{L^2(0,H_1)}}{|x_2-y_2|^2}\,dx_2dy_2 .$$

Then

$$(4.6)\quad \|v\|_{H^{1/2}(F)} = \Big(|v|^2_{H^{1/2}(F)} + \frac{1}{H}\|v\|^2_{L^2(F)}\Big)^{1/2}$$

for $v \in H^{1/2}(F)$ and

$$(4.7)\qquad \|v\|_{H_{00}^{1/2}(F)} = \left(|v|^2_{H^{1/2}(F)} + I(v;H_1,H_2)\right)^{1/2}$$

for $v \in H_{00}^{1/2}(F)$ where

$$I(v;H_1,H_2) = \int_0^{H_1} \frac{\|v(x_1,\cdot)\|^2_{L^2(0,H_2)}}{x_1}\,dx_1 + \int_0^{H_1} \frac{\|v(x_1,\cdot)\|^2_{L^2(0,H_2)}}{H_1-x_1}\,dx_1 + \int_0^{H_2} \frac{\|v(\cdot,x_2)\|^2_{L^2(0,H_1)}}{x_2}\,dx_2 + \int_0^{H_2} \frac{\|v(\cdot,x_2)\|^2_{L^2(0,H_2)}}{H_2-x_2}\,dx_2$$

LEMMA 2. Let Ω_i be a cube or tetrahedron with faces F_{ij} . Let f_i be equal to f_{ij} on a fixed face F_{ij} and zero on the remaining faces as well as on ∂F_{ij} and let $f_{ij} \in H_{00}^{1/2}(F_{ij})$. Then there exists an extension u of f_i on Ω_i such that $u = f_i$ on $\partial\Omega_i$ and

$$(4.8)\qquad |u|^2_{H^1(\Omega_i)} \leqq c\,\|f_{ij}\|^2_{H_{00}^{1/2}(F_{ij})}$$

Proof. We only give a proof for a cube and the case when $H = 1$. The proof of the other cases are quite similar.

In view of the extension theorem, see [9], there exists an extension u of f_i such that $u = f_i$ on $\partial\Omega_i$ and

$$(4.9)\qquad \|u\|_{H^1(\Omega_i)} \leqq c\,\|f_i\|_{H^{1/2}(\partial\Omega_i)}$$

By the definition of the norm in $H^{1/2}(\partial\Omega_i)$

$$\|f_i\|^2_{H^{1/2}(\partial\Omega_i)} =$$

(4.10)

$$= \int_{\partial\Omega_i}\int_{\partial\Omega_i} \frac{|f_i(x)-f_i(y)|^2}{|x-y|^3}\,dxdy + \|f_i\|^2_{L^2(\partial\Omega_i)}$$

But

$$\int_{\partial\Omega_i}\int_{\partial\Omega_i} \frac{|f_i(x)-f_i(y)|^2}{|x-y|^3}\,dxdy =$$

$$= \int_{F_{i1}}\int_{F_{i1}} \frac{|f_{i1}(x)-f_{i1}(y)|^2}{|x-y|^3}\,ds(x)ds(y) +$$

$$+ \sum_{j\neq 1} \int_{F_{i1}}\int_{F_{ij}} \frac{|f_{i1}(x)|^2}{|x-y|^3}\,ds(x)ds(y)$$

where $s(x)$ is the area element along F_{i1}.
We assume that f_i differs from zero on the face F_{i1}.
Using the inequality (1,3,2,12) from [7] we obtain

$$\sum_{j\neq 1} \int_{F_{i1}}\int_{F_{ij}} \frac{|f_{i1}(x)|^2}{|x-y|^3}\,ds(x)ds(y) \leqq$$

$$\leqq C \int_{F_{i1}} \frac{(f_{i1}(x))^2}{d(x,\partial F_{i1})}\,ds(x)$$

It is easy to see that

$$\|f_i\|^2_{L^2(\partial\Omega_i)} = \|f_i\|^2_{L^2(F_{i1})} \leq C \int_{F_{i1}} \frac{(f_{i1}(x))^2}{d(x,\partial F_{i1})}\, ds(x)$$

Substituting these inequalities into (4.10), we obtain

$$\|f_i\|^2_{H^{1/2}(\partial\Omega_i)} \leq C \|f_{i1}\|^2_{H^{1/2}_{00}(F_{i1})}$$

which proves (4.8) in view of (4.9) and Friedrichs' inequality.

We next consider a similar question for finite element functions.

LEMMA 3. Let Ω_i be a cube or tetrahedron with faces F_{ij}. Let f_{ih} be a finite element function belonging to $V_h^{(1)}(\partial\Omega_i)$ which is equal to f_{ijh} on the fixed face F_{ij} and zero on the remaining faces as well as on ∂F_{ij}. Then there exists an extension $u_h \in V_h^{(1)}$ of f_{ih} onto Ω_i such that $u_h = f_{ih}$ on $\partial\Omega_i$ and

$$(4.11)\quad |u_h|^2_{H^1(\Omega_i)} \leq C(1+\log\tfrac{H}{h})^2 \|f_{ijh}+\alpha\|^2_{H^{1/2}(F_{ij})}$$

where α is any constant.

Proof. It suffices to prove (4.11) under the assumption that $H = 1$. We will also only give a proof for the case of a cube.

Suppose that f_{ih} differes from zero on the face F_{i1} belonging to the (x_1,x_2)-plane. As a consequence of Lemma 2 there exists $\tilde{u} \in H^1(\Omega_i)$ such that $\tilde{u} = f_{ih}$ on $\partial\Omega_i$ and

$$|\tilde{u}|^2_{H^1(\Omega_i)} \leq C \|f_{i1}\|^2_{H^{1/2}_{00}(F_{i1})}$$

However in general $\tilde{u} \notin V_h^{(1)}(\Omega_i)$. Using an extension theorem for the finite element functions, see [10], we can construct $u_h \in V_h^{(1)}(\Omega_i)$ such that $u_h = \tilde{u}$ on $\partial\Omega_i$ and

$$|u_h|^2_{H^1(\Omega_i)} \leqq C|\tilde{u}|^2_{H^1(\Omega_i)}$$

Hence

$$(4.12) \qquad |u_h|^2_{H^1(\Omega_i)} \leqq C\,\|f_{i1}\|^2_{H^{1/2}_{00}(F_{i1})}$$

We now estimate the right hand side of (4.12). By (4.7),

$$\|f_{i1}\|^2_{H^{1/2}_{00}(F_{i1})} \leqq C(|f_{i1}|^2_{H^{1/2}(F_{i1})} + I(f_{i1};1,1))$$

Denote the first term of $I(f_{i1};1,1)$ by $I_1(f_{i1})$. Let Δ_i $i=1,\ldots,p$, denote triangles with one or two vertices on $x_1 = 0$. Let d_i be the distance to the x_2 axis from a vertex of Δ_i, which does not belong to $x_1 = 0$, and is closer to $x_1 = 0$. We represent $I_1(f_{i1})$ as

$$I_1(f_{i1}) = \int_0^d \frac{\|f_{i1}(x_1,\cdot)\|^2_{L^2}}{x_1}\,dx_1 + \int_d^1 \frac{\|f_{i1}(x_1,\cdot)\|^2_{L^2}}{x_1}\,dx_1 =$$

$$= I_{11}(f_{i1}) + I_{12}(f_{i2})$$

where $d = \inf d_i$. Using for each Δ_i the mean value theorem, the inverse inequality and the fact that the triangulation of F_{ij} is regular, we straightforwardly show that

$$I_{11}(f_{i1}) \leqq C \max_{0\leqq x_1\leqq h} \|f_{i1}(x_1,\cdot)\|^2_{L^2(0,1)} \quad .$$

It is also easy to see that

$$I_{12}(f_{i2}) \leqq C\,\log(\tfrac{H}{h}) \max_{d\leqq x_1\leqq 1} \|f_{i1}(x_1,\cdot)\|^2_{L^2(0,1)}$$

Hence

$$I_1(f_{i1}) \leq C(1+\log\frac{H}{h}) \max_{x_1} \| f_{i1}(x_1,\cdot)\|^2_{L^2(0,1)}$$

We now apply Lemma 1 with respect to x_1 and obtain

$$I_1(f_{i1}) \leq C(1+\log\frac{H}{h})^2 \int_0^1 \| f_{i1}(\cdot,x_2) + \alpha\|^2_{H^{1/2}(0,1)} dx_2 \leq$$

$$\leq C(1+\log\frac{H}{h})^2 \| f_{i1}+\alpha\|^2_{H^{1/2}(F_{i1})}$$

where α is any constant. In the similar way we estimate the remaining terms of $I(f_{i1};1,1)$. We thus obtain, by addition,

$$\| f_{i1}\|^2_{H^{1/2}_{00}(F_{i1})} \leq C(1+\log\frac{H}{h})^2 \| f_{i1}+\alpha\|^2_{H^{1/2}(F_{ij})}$$

which proves (4.11) for a cube.

Let E_{ih} denote the set of nodal points belonging to the wire basket E_i of Ω_i.

LEMMA 4. Let α_i be the average value of v on a cube or tetrahedron Ω_i. Then for any $v \in V_h^{(1)}(\Omega_i)$

$$h \sum_{x\in E_{ih}} (v(x) - \alpha_i)^2 \leq C(1+\log\frac{H}{h})|v|^2_{H^1(\Omega_i)} \tag{4.13}$$

holds.

Proof. It is easy to see that

$$h \sum_{x\in E_{ih}} (v(x) - \alpha_i)^2 \leq C \sum_j \int_{E_{ij}} (v(x)-\alpha_i)^2 ds(x) \tag{4.14}$$

Let Ω_i be a cube with a face $F_{ij} = (0,H)^2$ in the (x_1,x_2)-plane. Then

$$\int_{E_{ij}} (v(x)-\alpha_i)^2 dx \leq C \max_{x_1} \int_0^H (v(x)-\alpha_i)^2 dx_2$$

Applying Lemma 1 we get

$$\int_{E_{ij}} (v(x) - \alpha_i)^2 dx \leqq C(1+\log\frac{H}{h}) \| v - \alpha_i \|^2_{H^{1/2}(F_{ij})}$$

Further

$$\| v - \alpha_i \|^2_{H^{1/2}(F_{ij})} \leqq C |v|^2_{H^1(\Omega_i)}$$

in view of the trace theorem, see [9] , and Poincaré's inequality. Thus

$$\int_{E_{ij}} (v(x) - \alpha_i)^2 dx \leqq C(1+\log\frac{H}{h}) |v|^2_{H^1(\Omega_i)} \quad .$$

Substituting this into (4.14) we get (4.13) for a cube.
The proof for a tetrahedron is quite similar.

5. Preconditioners. We represent the bilinear form $a(u,v)$ defined for $u,v \in V_h^{(1)}$ as

$$a(u,v) = a_D(u,v) + a_N(u,v)$$

where $a_D(u,v)$ and $a_N(u,v)$ are the restrictions of $a(u,v)$ to Ω_D and Ω_N.

We introduce an auxiliary function $\hat{I}_H v$ as a global harmonic extension of v from the wire basket E as follows.
We rewrite the matrix A, see (3.1), as a 2x2 block matrix of the form

$$A = \begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} \\ \hat{A}_{12}^T & \hat{A}_{22} \end{bmatrix}$$

Here the matrices $\hat{A}_{11}$ and $\hat{A}_{22}$ represent the couplings between pairs of degree of freedom in $\Omega \setminus E$ and E respectively. The function $\hat{I}_H v$ is defined as the solution of

$$\hat{A}_{11}\hat{v}_1 + \hat{A}_{12}\hat{v}_2 = 0$$

with $\hat{v}_2$ given on E. In other words $\hat{I}_H v$ is the discrete harmonic extension (with respect to $\hat{A}_{11}$) of v given on E.

A preliminary preconditioner for $a(v,v)$ is introduced as

$$b(v,v) = a_D(v-\hat{I}_H v, v-\hat{I}_H v) + a_N(v-\hat{I}_H v, v-\hat{I}_H v) + a(\hat{I}_H v, \hat{I}_H v)$$

Note that

$$a(\hat{I}_H v, \hat{I}_H v) \leqq a(w,w)$$

for arbitrary $w \in V_h^{(1)}(\Omega)$ equal to $\hat{I}_H v$ on E since the function $\hat{I}_H v$ is the discrete harmonic function and therefore the minimal extension of the data on E. Using this fact and the triangle inequality we get

$$\frac{1}{5}\, b(v,v) \leqq a(v,v) \leqq 2\, b(v,v)$$

We will modify the bilinear form $b(v,v)$. From now on, we will consider only the case when $f_1 = 0$ in the system (3.1).
Such a reduction can be accomplished at the expense of solving the Dirichlet subproblems on Ω_D. This means that the first component u_1 of the solution $u=(u_1,u_2,u_3)^T$ is the discrete harmonic function in Ω_D (with respect to A_{11}), see (3.1).

We now show that

$$(5.1) \qquad a_D(v-\hat{I}_H v, v-\hat{I}_H v) \leqq C(1+\log\tfrac{H}{h})^2 a_N(v-\hat{I}_H v, v-\hat{I}_H v)$$

under the assumption that v is a discrete harmonic function in Ω_D. Let $w = v - \hat{I}_H v$. Note that w is also a discrete harmonic function in Ω_D. This implies that

$$(5.2) \qquad a_D(w,w) \leqq a_D(\tilde{w},\tilde{w})$$

for arbitrary $\tilde{w} \in \underline{V}_h^{(1)}(\Omega)$ restricted to Ω_D and equal to w on $\overline{\Omega}_N$. Furthermore

$$(5.3) \qquad a_D(\tilde{w},\tilde{w}) \leqq C(|\tilde{w}|^2_{H^1(\Omega_D)} + a_N(w,w))$$

By definition,

$$(5.4) \qquad |\tilde{w}|^2_{H^1(\Omega_D)} = \sum_{D\text{-type}} |\tilde{w}|^2_{H^1(\Omega_i)}$$

Choosing $\tilde{w}$ according to Lemma 3 we get

$$|\tilde{w}|^2_{H^1(\Omega_i)} \leq C(1+\log\frac{H}{h})^2 \sum_j |w-\alpha_j|^2_{H^{1/2}(F_{ij})} .$$

Using the trace theorem, see [9] , and Poincaré inequality we obtain

$$|\tilde{w}|^2_{H^1(\Omega_i)} \leq C(1+\log\frac{H}{h})^2 \sum_j |w|^2_{H^1(\Omega_j)}$$

where Ω_j denotes a N-type subregion with face $F_{ij} = \bar{\Omega}_i \cap \bar{\Omega}_j$. Summing these estimates with respect to i and substituting the resulting inequality into (5.4) we get

$$|\tilde{w}|^2_{H^1(\Omega_D)} \leq C(1+\log\frac{H}{h})^2 |w|^2_{H^1(\Omega_N)}$$

This implies (5.1) in view of (5.3) and (5.2).

We return to the preliminary preconditioner $b(v,v)$. Using (5.1) we see that

$$c(v,v) = a_N(v-\hat{I}_H v, v-\hat{I}_H v) + a(\hat{I}_H v, \hat{I}_H v)$$

can be used as a preconditioner for $a(v,v)$. It is easy to see that

$$\frac{1}{5}c(v,v) \leq a(v,v) \leq C(1+\log\frac{H}{h})^2 c(v,v)$$

However when using this preconditioner one needs to compute $\hat{I}_H v$ which is an expansive procedure. Therefore we further modify $c(v,v)$. We will discuss two cases. In the first case Ω consists of a few subregions and in the other case we have a large number.

We start with the first case. Let

$$\hat{B}_E(v,v) = \sum_i h \sum_{E_{ih}} v^2(x)$$

where E_{ih} is the set of nodal points on E_i . Applying Lemma 4 and using the fact that $\hat{I}_H v$ is a discrete harmonic extension from E , we show that

$$(5.5) \qquad C_0 \delta^{-1} \hat{B}_E(v,v) \leq a(\hat{I}_H v, \hat{I}_H v) \leq C_1 \hat{B}_E(v,v)$$

where $\delta = (1 + H^{-2})(1 + \log(H/h))$.

We now introduce a function $I_H v$ as a particular local harmonic extension (with respect to $a_N(v,w)$ and $a_D(v,w)$) of v from E_i. It is defined as follows. First we extend v given on E_i to N-type regions Ω_i by solving the mixed problems with Neumann boundary conditions on the faces of Ω_i. Then we extend the result to D-type regions by solving the obvious Dirichlet problems.

Using the function $I_H v$ we have

$$a_N(v-\hat{I}_H v, v-\hat{I}_H v) \leqq 2a_N(v-I_H v, v-I_H v) + C_2 \hat{B}_E(v,v)$$

in view of (5.5) and the fact that $I_H v$ is minimal extension in N-type regions of the data on E_i. Thus we can take

$$d(v,v) = a_N(v-I_H v, v-I_H v) + \hat{B}_E(v,v)$$

as a preconditioner for $a(v,v)$. We have proved the following result

THEOREM 1. Let the triangulation Ω_H and Ω_h of Ω be regular. Then for the discrete harmonic functions in Ω_D

$$C_0 \delta^{-1} d(v,v) \leqq a(v,v) \leqq C_1(1+\log\tfrac{H}{h})^2 d(v,v)$$

holds, where $\delta = (1 + H^{-2})(1 + \log(H/h))$.

We now describe briefly an algorithm for solving the resulting system corresponding to $d(v,v)$:

$$d(u,v) = G(v), \qquad v \in V_h^{(1)}(\Omega) .$$

ALGORITHM 1.

1. Find $w = u - I_H u$ by solving

$$a_N(w, \varphi_k) = G(\varphi_k)$$

where φ_k are the natural basis functions of $V_h^{(1)}(\Omega)$ associated with nodal points $x \in \bar{\Omega}_N \setminus E$. Note that $w=0$ on E and $a_N(w, I_H \varphi_k) = 0$ by the definition of $I_H v$. We find the solution of this system by solving the subproblems on individual N-type regions with a homogeneous Dirichlet boundary condition on the wire basket E. These subproblems are independent and may be done in parallel.

2. Solve

$$\hat{B}_E(u,\varphi_k) = G(\varphi_k) - a_N(w,\varphi_k)$$

where the φ_k are basis functions associated with $x \in E$.

3. Solve

$$a_N(I_H u,\varphi_k) = 0$$

where the φ_k are associated with $x \in \bar{\Omega}_N \setminus E$. Here $I_H u = u$ is given on E.

4. Compute $u = w + I_H u$.

In an entire algorithm for solving the problem we need one more point, i.e. an extension u to the D-type region to compute the residual vector.

We now consider the case with a large number of subregions. Two preconditioners will be described. Let

$$B_E(v,v) = \sum_i h \sum_{E_{ih}} (v(x) - \bar{v}_i)^2$$

where $\bar{v}_i$ is the discrete average value of v on E_{ih}, i.e.

$$\bar{v}_i = \frac{1}{n_i} \sum_{E_{ih}} v(x)$$

and n_i is the number of nodal points on E_i. Note that

$$\sum_{E_{ih}} (v(x) - \bar{v}_i)^2 \leqq \sum_{E_{ih}} (v(x) - \alpha_i)^2$$

for any constant α_i. Using this, Lemmas 4 and 1, and replacing $\hat{I}_H v$ by $I_H v$ we can prove that

$$C_2(1+\log\tfrac{H}{h})^{-1}B_E(v,v) \leqq a(\hat{I}_H v,\hat{I}_H v) \leqq C_3(1+\log\tfrac{H}{h})^2 B_E(v,v)$$

cf. (5.5). Thus we have a new preconditioner for $a(v,v)$ of the form

$$e(v,v) = a_N(v-I_H v, v-I_H v) + (1+\log\tfrac{H}{h})^{-1}B_E(v,v)$$

THEOREM 2. Under the assumption of Theorem 1,

$$C_4 e(v,v) \leqq a(v,v) \leqq C_5(1+\log\tfrac{H}{h})^3\, e(v,v)\,.$$

An algorithm for solving the equation

$$e(u,v) = G(v) , \quad v \in V_h^{(1)}(\Omega)$$

differs from Algorithm 1 in the second step only.

ALGORITHM 2

1. The same as in Algorithm 1
2. a) Find $\hat{w} = u(x) - \bar{u}_1$ by solving

$$B_E(\hat{w}, \varphi_k) = (G(\varphi_k) - a_N(w, \varphi_k))(1+\log\frac{H}{h})$$

where the φ_k are associated with $x \in E$.

b) Compute $\bar{u}_1$ using the algorithm suggested recently in [3] .

c) Compute $u(x) = \hat{w} + \bar{u}_1$ on E_1 .

3. The same as in Algorithm 1 .
4. Compute $u = w + I_H u$.

Finally we consider a preconditioner with piecewise linear functions on the coarse triangulation. Let $\tilde{I}_H v$ be the piecewise linear interpolant of v using the vertices of the coarse triangulation Ω_H . Note that $\tilde{I}_H v$ is a discrete harmonic extension from E .

We introduce a bilinear form

$$g(v,v) = a_N(v-I_H v, v-I_H v) + \tilde{B}_E(v,v) + a(\tilde{I}_H v, \tilde{I}_H v)$$

where

$$\tilde{B}_E(v,v) = h \sum_I \sum_{E_{1h}} (v(x) - \tilde{I}_H v)^2$$

Using Lemma 1 and Lemma 4 and the properties of the functions $I_H v$ and $\tilde{I}_H v$ one can be proved the following estimates:

$$\|\tilde{I}_H v\|^2_{L^2(0,H)} \leqq C(1+\log\frac{H}{h})(\|v\|^2_{L^2(0,H)} + H|v|^2_{H^{1/2}(0,H)})$$

and

$$a(\tilde{I}_H v, \tilde{I}_H v) \leqq C\,\frac{H}{h}(1+\log\frac{H}{h})a(v,v)$$

Using these we can prove the following result

THEOREM 3. Under the assumptions of Theorem 1

$$C_4 \delta_1^{-1} g(v,v) \leqq a(v,v) \leqq C_5(1+\log\frac{H}{h})^2 g(v,v)$$

holds, $\delta_1 = (H/h)(1+\log(H/h))$.

This result shows that this type of algorithm is far from optimal if H/h is large. We note that the corresponding algorithm works quite well for problems in the plane, cf. [3].

An algorithm for solving

$$g(u,v) = G(v)$$

takes the form

ALGORITHM 3

1. The same as in Algorithm 1.
2. Find $\hat{w} = u - \tilde{I}_H u$ by solving

$$\tilde{B}_E(\hat{w}, \varphi_k) = (G(\varphi_k) - a_N(w, \varphi_k))$$

where the φ_k are associated with $x \in E \setminus P$ and P is the set of the vertices of the subdomains.

3. Solve

$$a(\tilde{I}_H u, \tilde{I}_H \varphi_k) = G(\varphi_k) - a_N(w, \varphi_k) - \tilde{B}_E(\hat{w}, \varphi_k - \tilde{I}_H \varphi_k)$$

where the φ_k are associated with $x \in P$.

4. Compute $u = \hat{w} + \tilde{I}_H u$ on E_i.
5. Solve $a_N(I_H u, \varphi_k) = 0$

where the φ_k are associated with $x \in \bar{\Omega}_N \setminus E$.

6. Compute $u = w + I_H v$.

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