Adaptive BDDC and FETI-DP methods with change of basis formulation

Hyea Hyun Kim[1], Eric T. Chung[2], and Junxian Wang[3]

1 Introduction

In this paper, BDDC (Balancing Domain Decomposition by Constraints) and FETI-DP (Dual-Primal Finite Element Tearing and Interconnecting) algorithms with a change of basis for adaptive primal constraints are analyzed. In our formulation, adaptive primal constraints are introduced from appropriate generalized eigenvalue problems. In the authors previous study Kim et al. [2017a], for the FETI-DP algorithm the adaptive primal constraints are enforced by using a projection and it was shown that the condition numbers are controlled by the user-defined tolerance value, which is used to select the adaptive primal constraints from generalized eigenvalue problems on each equivalence classes, edges and faces. The analysis in Kim et al. [2017a] could not be extended to the FETI-DP algorithm with a change of basis formulation on the adaptive primal constraints. In the change of basis formulation, each primal constraint is transformed into a single unknown and treated just like unknowns at subdomain vertices as in the standard FETI-DP algorithm. It is often observed that the change of basis formulation is numerically more stable than the projection approach.

Here we will propose a more general form of the FETI-DP preconditioner and extend the analysis to the change of basis formulation. For the proposed preconditioner, we can obtain the identity $E_D + P_D = I$ for the averaging and jump operators , see (8) for their definitions, and thus show that the condition numbers of the adaptive BDDC and FETI-DP algorithms with the change of basis formulation are identical. Unlike in the standard FETI-DP preconditioners, the blocks of subdomain matrices and scaling matrices corresponding to the adaptive primal unknowns appear in the proposed preconditioner. We

[1]Department of Applied Mathematics and Institute of Natural Sciences, Kyung Hee University, Korea. hhkim@khu.ac.kr ·[2]Department of Mathematics, The Chinese University of Hong Kong, Hong Kong SAR. tschung@math.cuhk.edu.hk ·[3]School of Mathematics and Computational Science, Xiangtan University, China. wangjunxian@xtu.edu.cn

note that in the same mini-symposium an adaptive FETI-DP algorithm with a change of basis formulation was presented in the talk by Axel Klawonn, where different generalized eigenvalue problems are introduced and different tools are used in the analysis of condition numbers.

We note that adaptive primal constraints are often required to obtain robustness of domain decomposition preconditioners with respect to coefficient variations in the model problem. For related works, we refer to Galvis and Efendiev [2010] and Dolean et al. [2012] for two-level additive Schwarz methods, and Spillane et al. [2013] and Spillane and Rixen [2013] for FETI/BDD methods. In a pioneering work by Mandel et al. [2012], adaptive BDDC algorithms are developed and tested for $3D$ problems, where the adaptive primal constraints are selected from generalized eigenvalue problems on each face. For $3D$ problems, more advanced FETI-DP/BDDC algorithms are developed and analyzed in more recent works, see Klawonn et al. [2016],Calvo and Widlund [2016], and Kim et al. [2017b]. In Klawonn et al. [2016], Kim et al. [2017b], and Kim et al. [2017a], the adaptive primal constraints are enforced by using a projection in the FETI-DP algorithm.

2 BDDC and FETI-DP algorithms

For the presentation of BDDC and FETI-DP algorithms, we introduce a finite element space X for a given domain Ω, where the model elliptic problem is define as

$$-\nabla \cdot (\rho(x)\nabla u(x)) = f(x) \tag{1}$$

with a zero boundary condition on $u(x)$ and with $\rho(x)$ being highly varying and heterogeneous. The domain Ω is then partitioned into non-overlapping subdomains $\{\Omega_i\}$. We assume that the subdomain boundaries do not cut the triangles in the finite element space X. We use the notation X_i to denote the restriction of X to Ω_i. Each subdomain is then equipped with the finite element space X_i.

We further introduce W_i as the restriction of X_i to the subdomain interface unknowns, W, and X as the product of local finite element spaces W_i and X_i, respectively. We note that functions in W or X are decoupled across the subdomain interfaces. We then select some primal unknowns among the decoupled unknowns on the interfaces and enforce continuity on them and denote the corresponding spaces $\widetilde{W}$ and $\widetilde{X}$.

The preconditioners in BDDC and FETI-DP algorithms will be developed based on the partially coupled space $\widetilde{W}$ and appropriate scaling matrices. In our adaptive methods, we will select primal unknowns on each nodal equivalence classes of subdomain interfaces. In more detail, edges in $2D$ and faces in $3D$ are nodal equivalence classes shared by two subdomains, edges in $3D$ are nodal equivalence classes shared by more than two subdomains, and vertices are end points of edges in both $2D$ and $3D$.

In our approach, we first include the unknowns at subdomain vertices to the set of primal unknowns. Adaptive primal constraints will be selected from eigenvectors of generalized eigenvalue problems on faces and edges using a given tolerance value. The associated adaptive primal unknowns are then obtained by applying change of basis on the adaptively selected primal constraints and these explicit unknowns can then be assembled strongly just like unknowns at subdomain vertices.

We introduce notations K_i and S_i. The matrices K_i are obtained from Galerkin approximation of

$$a(u,v) = \int_{\Omega_i} \rho(x) \nabla u \cdot \nabla v \, dx$$

by using finite element spaces X_i and S_i are Schur complements of K_i, which are obtained after eliminating unknowns interior to Ω_i. Let $\widetilde{R}_i : \widetilde{W} \rightarrow W_i$ be the restriction into $\partial\Omega_i$ and let $\widetilde{S}$ be a partially coupled matrix defined by

$$\widetilde{S} = \sum_{i=1}^{N} \widetilde{R}_i^T S_i \widetilde{R}_i. \tag{2}$$

We note that $\widetilde{S}$ is then coupled at the unknowns on subdomain vertices and the adaptive primal unknowns. Let $\widetilde{R}$ be the restriction from $\widehat{W}$ to $\widetilde{W}$, where the subspace $\widehat{W}$ of $\widetilde{W}$ has unknowns continuous on the subdomain interface. The discrete problem of (1) is then written as

$$\widetilde{R}^T \widetilde{S} \widetilde{R} = \widetilde{R}^T \widetilde{g},$$

where $\widetilde{g}$ is the vector related to the right hand side $f(x)$.

In the BDDC algorithm the above matrix equation is solved iteratively by using the following preconditioner,

$$M_{BDDC}^{-1} = \widetilde{R}^T \widetilde{D} \widetilde{S}^{-1} \widetilde{D}^T \widetilde{R}, \tag{3}$$

where $\widetilde{D}$ is a scaling matrix of the form

$$\widetilde{D} = \sum_{i=1}^{N} \widetilde{R}_i^T D_i \widetilde{R}_i.$$

Here the matrices D_i are defined for unknowns in W_i and they are introduced to resolve heterogeneity in $\rho(x)$ across the subdomain interface. In a more detail, D_i consists of blocks $D_F^{(i)}$, $D_E^{(i)}$, $D_V^{(i)}$, where F denotes corresponding blocks to faces, E to edges, and V to vertices, respectively. We note that those blocks satisfy the partition of unity for a given F, E, and V, respectively. We refer to Klawonn and Widlund [2006] for these definitions.

The FETI-DP algorithm is a dual form of the BDDC algorithm. After the change of unknowns on the adaptively selected constraints, we obtain the resulting FETI-DP algebraic system

$$B\widetilde{S}^{-1}B^T\lambda = d, \tag{4}$$

where $\widetilde{S}$ is the partially coupled matrix defined in (2), and B is the matrix with entries 0, -1, and 1, which is used to enforce continuity at the remaining decoupled interface unknowns, i.e., dual unknowns. We introduce the notation M for the set of Lagrange multipliers λ, of which dimension is identical to the number of continuity constraints enforced on the remaining decoupled interface unknowns. The above algebraic system is then solved by an iterative method with the following preconditioner

$$M_{FETI}^{-1} = \sum_{i=1}^{N} B_{D,\Delta}^{(i)} S_i (B_{D,\Delta}^{(i)})^T \tag{5}$$

where $(B_{D,\Delta}^{(i)})^T : M \rightarrow W_i$ is defined by

$$(B_{D,\Delta}^{(i)})^T \lambda|_F = D_{F,\Delta}^{(j)} \lambda_{ij} \text{ on each } F \in F(i) \tag{6}$$

and

$$(B_{D,\Delta}^{(i)})^T \lambda|_E = \sum_{l \in n(E,i)} D_{E,\Delta}^{(l)} \lambda_{il} \text{ on each } E \in E(i). \tag{7}$$

Here $F(i)$ and $E(i)$ denote the set of faces and edges of subdomain Ω_i, respectively, $n(E,i)$ denotes the set of neighboring subdomain indices sharing the edge E with Ω_i, and λ_{ij} denotes the part of Lagrange multipliers λ used to enforce continuity on the decoupled unknowns across Ω_i and Ω_j. The matrices $D_{F,\Delta}^{(j)}$ and $D_{E,\Delta}^{(l)}$ are given by blocks of $D_F^{(j)}$ and $D_E^{(l)}$ as follows,

$$D_{F,\Delta}^{(j)} = \begin{pmatrix} D_{F,\Delta\Delta}^{(j)} \\ D_{F,\Pi\Delta}^{(j)} \end{pmatrix}, \quad D_{E,\Delta}^{(l)} = \begin{pmatrix} D_{E,\Delta\Delta}^{(l)} \\ D_{E,\Pi\Delta}^{(l)} \end{pmatrix},$$

where the subscripts Δ and Π denote blocks of matrix $D_F^{(j)}$ and $D_E^{(l)}$ corresponding to the decoupled unknowns and the adaptive primal unknowns, respectively. For the unknowns at subdomain vertices, which belong to the initial set of primal unknowns, the values of $(B_{D,\Delta}^{(i)})^T\lambda$ are defined as zero. Differently from the standard FETI-DP preconditioner, the proposed preconditioner contains the scaling matrices involving the adaptive primal unknowns. With this new form of the FETI-DP preconditioner, we can show that the adaptive FETI-DP algorithm with the change of basis formulation has the same spectra except the values zero and one and thus can obtain the same condition number bound as that of the BDDC algorithm. When no adaptive primal unknowns are chosen, the preconditioner is identical to that considered in the standard FETI-DP algorithm.

3 Adaptively enriched coarse spaces

The adaptive constraints will be selected by considering generalized eigenvalue problems on each equivalence class. The idea is originated from the upper bound estimate of BDDC and FETI-DP preconditioner. In the estimate of condition numbers of BDDC and FETI-DP preconditioners, the average and jump operators are defined as

$$E_D = \widetilde{R}\widetilde{R}^T\widetilde{D}, \quad P_D = B_D^T B, \tag{8}$$

where $B = (B_\Delta \;\; 0)$ and $B_D^T = (B_{D,\Delta}^{(1)} \;\cdots\; B_{D,\Delta}^{(N)})^T$. We note that $B : \widetilde{W} \to M$ and $B_D^T : M \to W$, see the definition of $(B_{D,\Delta}^{(i)})^T$ in (6) and (7).

The adaptive constraints are then treated just like unknowns at subdomain vertices after change of basis formulation in both BDDC and FETI-DP algorithms, i.e., the continuity on them can be strongly enforced. We note that in our previous work one can not get $E_D + P_D = I$ when the standard FETI-DP preconditioner is considered for the change of basis formulation, i.e., without the blocks from the adaptive primal unknowns in the definition of the scaled jump operator B_D^T.

We will now introduce generalized eigenvalue problems for each face and each edge. For a face F, the following generalized eigenvalue problem is considered

$$A_F v_F = \lambda \widetilde{A}_F v_F, \tag{9}$$

where

$$A_F = (D_F^{(j)})^T S_F^{(i)} D_F^{(j)} + (D_F^{(i)})^T S_F^{(j)} D_F^{(i)}, \; \widetilde{A}_F = \widetilde{S}_F^{(i)} : \widetilde{S}_F^{(j)}.$$

In the above $S_F^{(i)}$ denote block matrix of S_i to the unknowns interior to F and $\widetilde{S}_F^{(i)}$ are Schur complements of S_i obtained by eliminating unknowns except those interior to F. The matrices then satisfy the following minimal energy property,

$$v_F^T \widetilde{S}_F^{(i)} v_F \le v^T S_i v, \text{ for any } v|_F = v_F, \tag{10}$$

where $v|_F$ denotes the restriction of v to the unknowns interior to F. The notation $A : B$ is a parallel sum defined as, see Anderson and Duffin [1969],

$$A : B = A(A+B)^+ B,$$

where $(A+B)^+$ denotes a pseudo inverse. The parallel sum satisfies the following properties

$$A:B = B:A, \quad A:B \leq A, \quad A:B \leq B, \tag{11}$$

and it was first used in forming generalized eigenvalues problems by Dohrmann and Pechstein [2013], of which idea was originated from the energy estimate of the average operator in the BDDC algorithm.

In (9), the eigenvalues are all positive and we select eigenvectors $v_{F,l},\ l \in N(F)$ with associated eigenvalues λ_l larger than the given λ_{TOL}. The following constraints will then be enforced on the unknowns in F,

$$(A_F v_{F,l})^T (w_F^{(i)} - w_F^{(j)}) = 0,\ l \in N(F).$$

After a change of basis, the above constraints can be transformed into explicit unknowns.

In $3D$, we can have an edge, a nodal equivalence class shared by more than two subdomains, and for an edge E we introduce the following generalized eigenvalue problem,

$$A_E v_E = \lambda \widetilde{A}_E v_E,$$

where

$$A_E = \sum_{m \in I(E)} \sum_{l \in I(E)\setminus\{m\}} (D_E^{(l)})^T S_E^{(m)} D_E^{(l)}, \quad \widetilde{S}_E = \prod_{m \in I(E)} \widetilde{S}_E^{(m)},$$

and $I(E)$ denotes the set of subdomain indices sharing E in common, and $\prod_{m \in I(E)} \widetilde{S}_E^{(m)}$ is the parallel sum of matrices $\widetilde{S}_E^{(m)}$. We note that $S_E^{(m)}$ and $\widetilde{S}_E^{(m)}$ are defined similarly as $S_F^{(m)}$ and $\widetilde{S}_F^{(m)}$. For a given λ_{TOL}, the eigenvectors with their eigenvalues larger than λ_{TOL} will be selected and denoted by $v_{E,l},\ l \in N(E)$. The following constraints will then enforced on the unknowns in E,

$$(A_E v_{E,l})^T (w_E^{(i)} - w_E^{(m)}) = 0,\ l \in N(E),\ m \in I(E) \setminus \{i\}.$$

Similarly to the face case, the above constraints can be transformed into explicit unknowns after the change of basis.

By using the adaptively selected primal unknowns on each face F and edge E as above, we can obtain the following estimate

$$\langle \widetilde{S}(I - E_D)\widetilde{w}, (I - E_D)\widetilde{w} \rangle \leq C \lambda_{TOL} \langle \widetilde{S}\widetilde{w}, \widetilde{w} \rangle, \tag{12}$$

where C is a constant depending on the maximum number of edges and faces per subdomain, and the maximum number of subdomains sharing an edge but independent of the coefficient $\rho(x)$. We note that the above inequality is the key estimate in the analysis of the BDDC algorithm.

4 Condition number estimate and numerical results

Using the adaptively enriched primal unknowns described in Section 3 and the estimate in (12), we can obtain the following estimate of condition numbers for the given λ_{TOL}:

Theorem 1. *The BDDC algorithm with the change of basis formulation for the adaptively chosen set of primal unknowns with a given tolerance λ_{TOL} has the following bound of condition numbers,*

$$\kappa(M_{BDDC}^{-1}\widetilde{R}^T\widetilde{S}\widetilde{R}) \leq C\lambda_{TOL},$$

and the FETI-DP algorithm with the change of basis formulation for the same set of adaptively chosen set of primal unknowns has the bound

$$\kappa(M_{FETI}^{-1}B\widetilde{S}^{-1}B^T) \leq C\lambda_{TOL},$$

where C is a constant depending only on $N_{F(i)}$, $N_{E(i)}$, $N_{I(E)}$, which are the number of faces per subdomain, the number of edges per subdomain, and the number of subdomains sharing an edge E, respectively. In fact, the two algorithms share the same set of eigenvalues except zero and one.

The proof of the above theorem and some numerical examples can be found in a complete version of this paper Kim et al. [2017c]. In Table 1, we present some numerical experiments for a 3D model problem. In particular, we consider a random coefficient with value varying between 10^{-3} to 10^3, and show the number of iterations and the number of primal unknowns with various choice of coarse partition N_d. We observe a very robust performance.

Table 1 Performance of adaptive BDDC and FETI-DP with $\lambda_{TOL}^F = 10, \lambda_{TOL}^E = 10^3$ for highly varying and random $\rho(x)$ in $(10^{-3}, 10^3)$ by increasing N_d and with a fixed $H/h = 12$: $\lambda_{\min}$ (minimum eigenvalues), $\lambda_{\max}$ (maximum eigenvalues), Iter (number of iterations), pnumF (total number of adaptive primal unknowns on faces), and pnumE (total number of adaptive primal unknowns on edges). pF and pE are the number of adaptive primal unknowns per face and per edge, respectively.

N_d	method	$\lambda_{\min}$	$\lambda_{\max}$	Iter	pnumF	pnumE	pF	pE
2^3	Bddc	1.00	5.29	18	21	18	1.75	3.00
	Fdp	1.00	5.29	18	21	18	1.75	3.00
3^3	Bddc	1.01	6.97	26	71	115	1.31	3.19
	Fdp	1.00	6.97	27	71	115	1.31	3.19
4^3	Bddc	1.01	9.45	29	205	320	1.42	2.96
	Fdp	1.00	9.45	30	205	320	1.42	2.96

Acknowledgements The first author was supported by the National Research Foundation of Korea(NRF) grants funded by NRF20151009350, the second author was supported by the Hong Kong RGC General Research Fund (Project 14317516) and the CUHK Direct

Grant for Research 2016-17, and the third author was supported by the National Natural Science Foundation of China (Grant No. 11201398) and Open Foundation of Guangdong Provincial Engineering Technology Research Center for Data Science(2016KF07).

References

W. N. Anderson, Jr. and R. J. Duffin. Series and parallel addition of matrices. *J. Math. Anal. Appl.*, 26:576–594, 1969.

Juan G. Calvo and Olof B. Widlund. An adaptive choice of primal constraints for BDDC domain decomposition algorithms. *Electron. Trans. Numer. Anal.*, 45:524–544, 2016.

Clark R. Dohrmann and Clemens Pechstein. Modern domain decomposition solvers: BDDC, deluxe scaling, and an algebraic approach, http://people.ricam.oeaw.ac.at/c.pechstein/pechstein-bddc2013.pdf. 2013.

Victorita Dolean, Frédéric Nataf, Robert Scheichl, and Nicole Spillane. Analysis of a two-level Schwarz method with coarse spaces based on local Dirichlet-to-Neumann maps. *Comput. Methods Appl. Math.*, 12(4):391–414, 2012.

Juan Galvis and Yalchin Efendiev. Domain decomposition preconditioners for multiscale flows in high-contrast media. *Multiscale Model. Simul.*, 8(4): 1461–1483, 2010.

Hyea Hyun Kim, Eric Chung, and Junxian Wang. BDDC and FETI-DP mathods with enriched coarse spaces for elliptic problems with oscillatory and high contrast coefficients. In *Domain decomposition methods in science and engineering XXIII*, volume 116 of *Lect. Notes Comput. Sci. Eng.*, pages 179–186. Springer, Heidelberg, 2017a.

Hyea Hyun Kim, Eric Chung, and Junxian Wang. BDDC and FETI-DP preconditioners with adaptive coarse spaces for three-dimensional elliptic problems with oscillatory and high contrast coefficients. *J. Comput. Phys.*, 349:191–214, 2017b.

Hyea Hyun Kim, Eric Chung, and Junxian Wang. Adaptive BDDC and FETI-DP algorithms with change of basis formulation. *Submitted.*, 2017c.

Axel Klawonn and Olof B Widlund. Dual-primal FETI methods for linear elasticity. *Comm. Pure Appl. Math.*, 59(11):1523–1572, 2006.

Axel Klawonn, Martin Kühn, and Oliver Rheinbach. Adaptive coarse spaces for FETI-DP in three dimensions. *SIAM J. Sci. Comput.*, 38(5):A2880–A2911, 2016.

Jan Mandel, Bedřich Sousedík, and Jakub Šístek. Adaptive BDDC in three dimensions. *Math. Comput. Simulation*, 82(10):1812–1831, 2012.

N. Spillane and D. J. Rixen. Automatic spectral coarse spaces for robust finite element tearing and interconnecting and balanced domain decomposition algorithms. *Internat. J. Numer. Methods Engrg.*, 95(11):953–990, 2013.

Nicole Spillane, Victorita Dolean, Patrice Hauret, Frédéric Nataf, and Daniel J. Rixen. Solving generalized eigenvalue problems on the interfaces to build a robust two-level FETI method. *C. R. Math. Acad. Sci. Paris*, 351(5-6):197–201, 2013.