Comparison of Continuous and Discrete Techniques to Apply Coarse Corrections

Martin J. Gander, Laurence Halpern, and Kévin Santugini-Repiquet

1 Introduction

There has been substantial attention on coarse correction in the domain decomposition community over the last decade, sparked by the interest of solving high contrast and multiscale problems, since in this case, the convergence of two-level domain decomposition methods is deteriorating when the contrast becomes large, see [1, 10, 16, 17, 11, 9, 8] and references therein. Our main interest here is not the content of the coarse spaces, but the way they are applied to correct the subdomain iterates. A classical way at the discrete level to apply coarse corrections, which led to the two level additive Schwarz method introduced in [2], is based on the residual like in multigrid: one computes the residual, projects it onto the coarse space, then solves a coarse problem which is for example obtained by a Galerkin projection of the fine system matrix on the coarse space, and then prolongates the correction by interpolation to the fine grid to add the correction to the current subdomain approximation. A complete analysis of this two level additive Schwarz preconditioner at the continuous level is given in [5], and for better coarse spaces, see [4, 6]. Another technique, also at the discrete level, is to use deflation, going back to the first coarse correction technique [15], where the functions spanning the coarse space are deflated, and then a deflated system is solved, see [14]. A further important class of coarse space correction techniques at the discrete level are the Balancing Domain Decomposition (BDD) methods [12, 13]. A more recent and very general approach at the continuous level for coarse correction is to approximately solve a

Martin J. Gander
Université de Genève, Switzerland e-mail: martin.gander@unige.ch

Laurence Halpern
Université Paris 13, France e-mail: halpern@math.univ-paris13.fr

Kévin Santugini-Repiquet
Bordeaux INP, IMB, UMR 5251, F-33400,Talence, France, e-mail: Kevin.Santugini-Repiquet@bordeaux-inp.fr

transmission problem for the error, as described in [7], which also shows that for domain decomposition methods discontinuous coarse spaces are of interest, since subdomain solutions are in general discontinuous in their traces and/or fluxes at the interfaces. This observation led to the DCS-DMNV algorithm (Discontinuous Coarse Space - Dirichlet Minimization and Neumann Variational) at the continuous level, a two-level iterative domain decomposition algorithm introduced in [3].

We are interested here in understanding if there is a relation between the coarse corrections formulated at the discrete level by a residual correction, like in Additive Schwarz, and the coarse correction obtained at the continuous level solving a transmission problem. These two approaches seem at first to be very different, and to be able to compare them, we will precisely compute the coarse correction one obtains with these two approaches for the very simple model problem

$$\mathcal{L}u := \partial_{xx}u = f \quad \text{in } \Omega := (0,1), \quad u(0) = u(1) = 0, \tag{1}$$

and two subdomain iterates u_j, $j = 1,2$ on the subdomains $\Omega_1 := (0, \frac{1}{2} + L)$ and $\Omega_2 := (\frac{1}{2} - L, 1)$ which were obtained by an arbitrary domain decomposition method, i.e. the subdomain iterates simply satisfy the equation in (1) and the outer homogeneous boundary conditions, but no other interface condition at $\frac{1}{2} - L$ and $\frac{1}{2} + L$. They can thus come from a Schwarz method if $L > 0$, optimized Schwarz method for both $L > 0$ and $L = 0$, or a FETI or Neumann-Neumann method if $L = 0$. To compare continuous and discrete techniques, we also assume that we have a discretization of (1) leading to a linear system of equations

$$A\mathbf{u} = \mathbf{f}, \tag{2}$$

and two discrete subdomain iterates $\mathbf{u}_j$, $j = 1,2$.

2 Discrete Coarse Correction Based on the Residual

Suppose our coarse space is spanned by two continuous functions q_1 and q_2, see for example the hat functions (thick solid blue lines) in Fig. 1. Evaluating them on the grid used for the discretization leads to two vectors $\mathbf{q}_1$ and $\mathbf{q}_2$. To formulate the classical residual based coarse correction like in multigrid and used in Additive Schwarz, one puts the two row vectors $\mathbf{q}_1$ and $\mathbf{q}_2$ into the coarse restriction matrix R_0, and forms the coarse matrix $A_0 := R_0 A R_0^T$, like in a classical Galerkin approach. Having two approximate discrete subdomain solutions $\mathbf{u}_1$ and $\mathbf{u}_2$, one forms a global approximation using a partition of unity χ_j (diagonal matrices summing to the identity in this discrete setting with ones on the diagonal outside the overlap),

$$\tilde{\mathbf{u}} := \chi_1 \mathbf{u}_1 + \chi_2 \mathbf{u}_2, \tag{3}$$

and then corrects this approximation by the residual correction formula

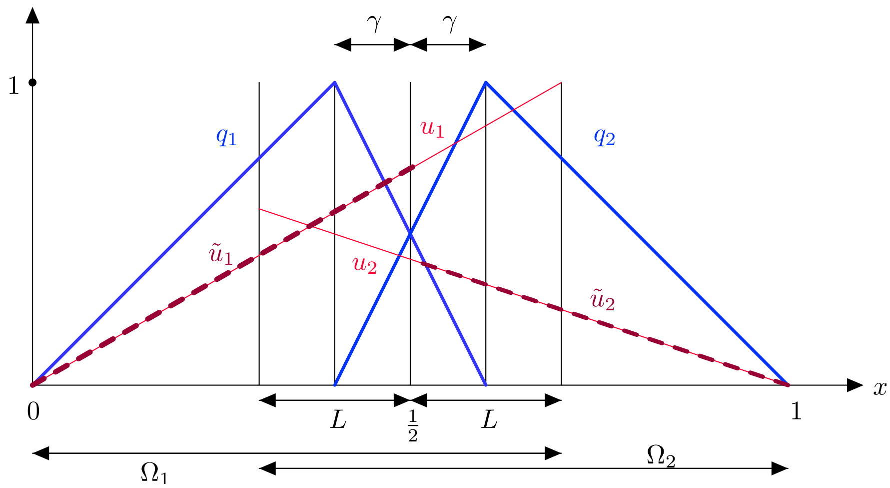

Fig. 1: Geometry with two subdomains Ω_j, coarse functions q_j and subdomain solutions u_j, $j = 1, 2$, which could be restricted to a non-overlapping decomposition to become $\tilde{u}_j$

$$\tilde{\mathbf{u}}^{new} := \tilde{\mathbf{u}} + R_0^T A_0^{-1} R_0(\mathbf{f} - A\tilde{\mathbf{u}}). \tag{4}$$

3 Continuous Coarse Correction Using a Transmission Problem

At the continuous level, a coarse correction can be computed by solving a transmission problem between the subdomains: for two approximate subdomain solution functions u_1 and u_2 shown as thin solid red lines in Figure 1, we restrict them first to a non-overlapping decomposition if the DD method used overlap,

$$\tilde{u}_1 := u_1|_{(0,\frac{1}{2})} \quad \text{and} \quad \tilde{u}_2 := u_2|_{(\frac{1}{2},1)}, \tag{5}$$

as shown with thick dashed dark red lines in Figure 1. If the DD method did not use overlap, we just denote by the tilde quantities $\tilde{u}_j$ the original iterates u_j, $j = 1, 2$. We then form the global approximation $\tilde{u}$ by gluing $\tilde{u}_1$ and $\tilde{u}_2$ together,

$$\tilde{u}(x) := \begin{cases} \tilde{u}_1(x) \text{ if } x \leq \frac{1}{2}, \\ \tilde{u}_2(x) \text{ if } x > \frac{1}{2}. \end{cases} \tag{6}$$

To compute the coarse correction, one can then for example use the DCS-DMNV technique, which we describe now using the coarse basis functions q_j shown with thick solid blue lines in Fig. 1 for the specific case when $\gamma = 0$: we define a continuous coarse space X_c and a discontinuous coarse space X_d by

$$X_c := \text{span}\{q_1 + q_2\}, \quad X_d := \text{span}\{q_1, q_2\}, \quad \gamma = 0. \tag{7}$$

Note that the glued solution $\tilde{u}$ lies in X_d. We then introduce a functional for measuring the jump in the approximate solution $\tilde{u}$ at the interfaces, which in our example would be at $x = \frac{1}{2}$,

$$q(v) := [v]^2(\frac{1}{2}), \quad [v](\frac{1}{2}) := v^+(\frac{1}{2}) - v^-(\frac{1}{2}). \tag{8}$$

To correct the approximation $\tilde{u}$, DCS-DMNV solves the minimization problem

$$\tilde{u}^{new} = \tilde{u} + \operatorname{argmin}_{v \in V} q(\tilde{u} + v) \tag{9}$$

over the constraint space

$$V := \{v \in X_d : \int_\Omega v'(x)w'(x)\,dx = [\tilde{u}'](\frac{1}{2})w(\frac{1}{2}),\ \forall w \in X_c\}. \tag{10}$$

The underlying vector space is

$$V_0 := \operatorname{span}(q_1 - q_2) \quad \text{and} \quad X_d = V_0 \oplus X_c. \tag{11}$$

4 Comparison of the Discrete and Continuous Techniques

In order to compare the discrete residual based coarse grid correction (4) to the continuous coarse correction (9) obtained by solving approximately a transmission problem using DCS-DMNV, we need to first formulate (4) at the continuous level for the specific case where the partition of unity (3) used in (4) glues the approximate subdomain solutions the same way as in DCS-DMNV which uses (6). Note that the glued function $\tilde{u}$ is a piece-wise C^∞ distribution, supported at $\frac{1}{2}$. This leads to

Theorem 1 *Let q_j for $j = 1, 2$ be the two hat functions in Figure 1,*

$$q_1 = \begin{cases} \frac{1}{\frac{1}{2}-\gamma}x & on\ (0, \frac{1}{2}-\gamma), \\ \frac{1}{2\gamma}(\frac{1}{2}+\gamma-x) & on\ (\frac{1}{2}-\gamma, \frac{1}{2}+\gamma), \\ 0 & on\ (\frac{1}{2}+\gamma, 1), \end{cases} \quad q_2 = \begin{cases} 0 & on\ (0, \frac{1}{2}-\gamma), \\ \frac{1}{2\gamma}(x-\frac{1}{2}+\gamma) & on\ (\frac{1}{2}-\gamma, \frac{1}{2}+\gamma), \\ \frac{1}{\frac{1}{2}-\gamma}(1-x) & on\ (\frac{1}{2}+\gamma, 1), \end{cases}$$

for $0 \le \gamma \le L$, and let the partition of unity (3) *be defined as in* (6)*. Then the continuous equivalent to the discrete residual based coarse correction* (4) *is*

$$\tilde{u}^{new} = \tilde{u} + (\frac{1}{2}-\gamma)\left((\frac{1}{2}[\tilde{u}'](\frac{1}{2}) + [\tilde{u}](\frac{1}{2}))q_1 + (\frac{1}{2}[\tilde{u}'](\frac{1}{2}) - [\tilde{u}](\frac{1}{2}))q_2\right). \tag{12}$$

Proof If u is a piece-wise C^2 function with a finite number of jumps at $a_1, \ldots, a_N$, and T_u denotes the distribution corresponding to u, we obtain for the derivatives using the jumps formula

$$T'_u = T_{u'} + \sum_{i=1}^{N} [u](a_i)\delta_{a_i}, \quad T''_u = T_{u''} + \sum_{i=1}^{N} ([u'](a_i)\delta_{a_i} + [u](a_i)\delta'_{a_i}).$$

Recall that the derivative of the Dirac distribution δ_a is defined by $\delta'_a(\phi) = -\phi'(a)$ for ϕ a C^1 function in the neighborhood of a. If we now apply the jump formula to $\tilde{u}$ we constructed by gluing the subdomain solutions together in (6), we obtain for the residual we need for the computation of the coarse correction

$$r := f - T''_{\tilde{u}} = -[\tilde{u}](\frac{1}{2})\delta'_{\frac{1}{2}} - [\tilde{u}'](\frac{1}{2})\delta_{\frac{1}{2}}.$$

The continuous equivalent to the discrete coarse correction (4) is to search for a coarse correction function $U = \alpha q_1 + \beta q_2$ such that

$$(T''_U, q_1) = (r, q_1), \quad (T''_U, q_2) = (r, q_2), \tag{13}$$

and to add it to $\tilde{u}$ to obtain $\tilde{u}^{new}$. We will work equivalently for this proof instead with the basis $(q_1 + q_2, q_1 - q_2)$ to solve system (13), which will naturally reveal the role played by the sum (continuous) and difference (discontinuous in the limit when γ goes to zero) and prepare for the relation with the DCS-DMNV approach. Working with the sum and difference also simplifies the solution of the system. We thus project now the residual r onto V_0 defined in (11) and X_c defined in (7), for which we need the functions $q_1 + q_2$ and $q_1 - q_2$,

$$q_1+q_2 = \begin{cases} \frac{1}{\frac{1}{2}-\gamma}x & \text{on } (0, \frac{1}{2} - \gamma), \\ 1 & \text{on } (\frac{1}{2} - \gamma, \frac{1}{2} + \gamma), \\ \frac{1}{\frac{1}{2}-\gamma}(1-x) & \text{on } (\frac{1}{2} + \gamma, 1), \end{cases} \quad q_1-q_2 = \begin{cases} \frac{1}{\frac{1}{2}-\gamma}x & \text{on } (0, \frac{1}{2} - \gamma), \\ \frac{1}{2\gamma}(1-2x) & \text{on } (\frac{1}{2} - \gamma, \frac{1}{2} + \gamma), \\ -\frac{1}{\frac{1}{2}-\gamma}(1-x) & \text{on } (\frac{1}{2} + \gamma, 1). \end{cases}$$

Since $q_1 + q_2$ is constant equal to 1 in $(\frac{1}{2} - \gamma, \frac{1}{2} + \gamma)$, we obtain

$$(r, q_1 + q_2) = -[\tilde{u}'](\frac{1}{2}), \quad (r, q_1 - q_2) = -\frac{1}{\gamma}[\tilde{u}](\frac{1}{2}).$$

We search now for a coarse correction $U = \alpha'(q_1 + q_2) + \beta'(q_1 - q_2)$ such that

$$(T''_U, q_1 + q_2) = (r, q_1 + q_2), \quad (T''_U, q_1 - q_2) = (r, q_1 - q_2). \tag{14}$$

From the jumps formula, we find

$$T''_{q_1 \pm q_2} = [q'_1 \pm q'_2](\frac{1}{2} - \gamma)\delta_{\frac{1}{2}-\gamma} + [q'_1 \pm q'_2](\frac{1}{2} + \gamma)\delta_{\frac{1}{2}+\gamma},$$

which leads to

$$T''_{q_1+q_2} = \frac{-1}{\frac{1}{2} - \gamma}(\delta_{\frac{1}{2}-\gamma} + \delta_{\frac{1}{2}+\gamma}), \quad T''_{q_1-q_2} = \frac{-1}{2\gamma(\frac{1}{2} - \gamma)}(\delta_{\frac{1}{2}-\gamma} - \delta_{\frac{1}{2}+\gamma}).$$

Since $(q_1+q_2)(\frac{1}{2}-\gamma) = (q_1+q_2)(\frac{1}{2}+\gamma) = 1$ and $(q_1-q_2)(\frac{1}{2}-\gamma) = -(q_1-q_2)(\frac{1}{2}+\gamma) = 1$, we find that

$$T''_{q_1+q_2}(q_1+q_2) = 2\gamma T''_{q_1-q_2}(q_1-q_2) = \frac{-2}{\frac{1}{2}-\gamma}, \quad T''_{q_1\pm q_2}(q_1 \mp q_2) = 0.$$

Inserting this into (14) gives a simple diagonal system for α' and β', namely

$$\frac{-2}{\frac{1}{2}-\gamma}\alpha' = -[\tilde{u}'](\frac{1}{2}), \quad \frac{-1}{\gamma(\frac{1}{2}-\gamma)}\beta' = -\frac{1}{\gamma}[\tilde{u}](\frac{1}{2}),$$

and thus for the coarse correction

$$U = (\frac{1}{2}-\gamma)(\frac{1}{2}[\tilde{u}'](\frac{1}{2})(q_1+q_2)+[\tilde{u}](\frac{1}{2})(q_1-q_2)),$$

which concludes the proof. □

For the DCS-DMNV algorithm for computing the coarse correction described in Section 3, we obtain the following theorem:

Theorem 2 *The coarse correction computed by the DCS-DMNV algorithm* (9)-(10) *is given by*

$$\tilde{u}^{new} = \tilde{u} + (\frac{1}{2}[\tilde{u}](\frac{1}{2}) + \frac{1}{4}[\tilde{u}'](\frac{1}{2}))q_1 + (-\frac{1}{2}[\tilde{u}](\frac{1}{2}) + \frac{1}{4}[\tilde{u}'](\frac{1}{2}))q_2,$$

which is equal to the limit of the coarse correction computed by the residual correction approach given in (12) *when γ goes to zero.*

Proof The DCS-DMNV algorithm uses the spaces V_0 and X_c, which we defined in (7) and (11) using the hat functions q_1 and q_2 for the specific case where $\gamma = 0$, in which q_1 and q_2 are discontinuous at $x = \frac{1}{2}$, and we have

$$q_1 + q_2 = \begin{cases} 2x & \text{on } [0, \frac{1}{2}], \\ 2(1-x) & \text{on } [\frac{1}{2}, 1], \end{cases} \qquad q_1 - q_2 = \begin{cases} 2x & \text{on } [0, \frac{1}{2}), \\ -2(1-x) & \text{on } (\frac{1}{2}, 1]. \end{cases}$$

We first note that X_c and V_0 are orthogonal subspaces of L^2, and the same holds for their derivatives, since $\|q_1\| = \|q_2\|$ and $\|q_1'\| = \|q_2'\|$. We next identify the constraint space V from (10): the function

$$v := \alpha'(q_1+q_2) + \beta'(q_1-q_2)$$

belongs to V if and only if

$$\int_\Omega (\alpha'(q_1'+q_2') + \beta'(q_1'-q_2'))(x)(q_1'+q_2')(x)\,dx = [\tilde{u}'](\frac{1}{2})(q_1+q_2)(\frac{1}{2}),$$

which gives

$$4\alpha' = [\tilde{u}'](\frac{1}{2}).$$

This defines V as the affine line

$$V = V_0 + \frac{1}{4}[\tilde{u}'](\frac{1}{2})(q_1 + q_2).$$

Therefore $U = \frac{1}{4}[\tilde{u}'](\frac{1}{2})(q_1 + q_2) + \beta'(q_1 - q_2)$. Now the Euler equation for (9) is

$$q'(\tilde{u} + U) \cdot v := 2[\tilde{u} + U](\frac{1}{2})[v](\frac{1}{2}) = 0 \; \forall v \in V_0. \tag{15}$$

Since $[q_1 - q_2](\frac{1}{2}) = 2$, (15) yields $[\tilde{u} + U](\frac{1}{2}) = 0$, and since $q_1 + q_2$ is continuous at $x = \frac{1}{2}$,

$$[\tilde{u}](\frac{1}{2}) + \beta'[q_1 - q_2](\frac{1}{2}) = [\tilde{u}](\frac{1}{2}) - 2\beta' = 0.$$

Therefore

$$U = \frac{1}{2}[\tilde{u}](\frac{1}{2})(q_1 - q_2) + \frac{1}{4}[\tilde{u}'](\frac{1}{2})(q_1 + q_2),$$

and we see that this is indeed the limit as $\gamma \to 0$ of the system (12). □

5 Conclusions

We have shown that two apparently quite different approaches for computing a coarse correction in domain decomposition, namely the residual based approach at the discrete level, and the approximate solution of a transmission problem at the continuous level using DCS-DMNV, lead to the same coarse correction in the limit when the discretized approach is computed at the continuous level, provided that one uses a discontinuous partition of unity. It therefore does not matter in this case which approach is used for computing the coarse correction, they are equivalent.

We showed our result for a simplified setting of Laplace's equation in 1D and for two subdomains only, but the generalization to many subdomains in 1D does not pose any difficulties, one just has to use the jumps formula several times. The generalization to higher spatial dimensions is also possible and not difficult in the case of strip decompositions. The case when cross points are present would however require more care and does not follow trivially. For a more general operator than the Laplacian, the generalization is in principle also possible, but one essential ingredient is that the coarse space functions q_j must satisfy the homogeneous equation, which is in general a desirable property for coarse space functions, see [7] and references therein.

A further open question is how the coarse correction computation based on deflation, and the BDD technique, are related to the two methods we compared here. We are currently studying these two techniques for the same simple model problem presented here, and also the higher dimensional case.

References

1. Aarnes, J., Hou, T.Y.: Multiscale domain decomposition methods for elliptic problems with high aspect ratios. Acta Math. Appl. Sin. Engl. Ser. **18**(1), 63–76 (2002)
2. Dryja, M., Widlund, O.B.: An additive variant of the Schwarz alternating method for the case of many subregions. Tech. Rep. 339, also Ultracomputer Note 131, Department of Computer Science, Courant Institute (1987)
3. Gander, M.J., Halpern, L., Santugini-Repiquet, K.: Discontinuous coarse spaces for DD-methods with discontinuous iterates. In: Domain Decomposition Methods in Science and Engineering XXI. Springer LNCSE (2013)
4. Gander, M.J., Halpern, L., Santugini-Repiquet, K.: A new coarse grid correction for RAS. In: Domain Decomposition Methods in Science and Engineering XXI. Springer LNCSE (2013)
5. Gander, M.J., Halpern, L., Santugini-Repiquet, K.: Continuous analysis of the additive Schwarz method: A stable decomposition in H1 with explicit constants. ESAIM: Mathematical Modelling and Numerical Analysis **49**(3), 713–740 (2015)
6. Gander, M.J., Halpern, L., Santugini-Repiquet, K.: New coarse corrections for Optimized Restricted Additive Schwarz using PETSc. In: Domain Decomposition Methods in Science and Engineering XXV. submitted (2018)
7. Gander, M.J., Halpern, L., Santugini-Repiquet, K.: On optimal coarse spaces for domain decomposition and their approximation. In: Domain Decomposition Methods in Science and Engineering XXIV. Springer LNCSE (2018)
8. Gander, M.J., Loneland, A.: SHEM: An optimal coarse space for RAS and its multiscale approximation. In: Domain Decomposition Methods in Science and Engineering XXIII. Springer (2016)
9. Gander, M.J., Loneland, A., Rahman, T.: Analysis of a new harmonically enriched multiscale coarse space for domain decomposition methods. arXiv preprint arXiv:1512.05285 (2015)
10. Graham, I.G., Lechner, P.O., Scheichl, R.: Domain decomposition for multiscale PDEs. Numer. Math. **106**(4), 589–626 (2007)
11. Klawonn, A., Radtke, P., Rheinbach, O.: FETI-DP methods with an adaptive coarse space. SIAM Journal on Numerical Analysis **53**(1), 297–320 (2015)
12. Mandel, J.: Balancing domain decomposition. Communications in Numerical Methods in Engineering **9**(3), 233–241 (1993). DOI:10.1002/cnm.1640090307
13. Mandel, J., Brezina, M.: Balancing domain decomposition for problems with large jumps in coefficients. Math. Comp. **65**, 1387–1401 (1996)
14. Nabben, R., Vuik, C.: A comparison of deflation and coarse grid correction applied to porous media flow. SIAM Journal on Numerical Analysis **42**(4), 1631–1647 (2004)
15. Nicolaides, R.A.: Deflation conjugate gradients with application to boundary value problems. SIAM J. Num. An, **24**(2), 355–365 (1987). DOI:doi:10.1137/0724027
16. Pechstein, C., Scheichl, R.: Analysis of FETI methods for multiscale PDEs. Numer. Math. **111**(2), 293–333 (2008)
17. Spillane, N., Dolean, V., Hauret, P., Nataf, F., Pechstein, C., Scheichl, R.: Abstract robust coarse spaces for systems of PDEs via generalized eigenproblems in the overlaps. Numerische Mathematik **126**(4), 741–770 (2014)