Parallel solver for $\boldsymbol{H}(\mathrm{div})$ problems using hybridization and AMG

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1 Introduction

This paper is concerned with the $\boldsymbol{H}(\mathrm{div})$ bilinear form acting on vector functions $\boldsymbol{u}$, $\boldsymbol{v}$:

$$a(\boldsymbol{u}, \boldsymbol{v}) = \int_{\Omega} \alpha \nabla \cdot \boldsymbol{u} \nabla \cdot \boldsymbol{v} + \beta \boldsymbol{u} \cdot \boldsymbol{v} \, d\boldsymbol{x}. \tag{1}$$

Here α, $\beta \in L^{\infty}(\Omega)$ are some positive heterogeneous coefficients, and Ω is a simply-connected polygonal domain in $\mathbb{R}^d$, $d = 2, 3$. Discrete problems associated with $a(\cdot, \cdot)$ arise in many applications, such as first order least squares formulation of second order elliptic problems (Cai et al. [1994]), preconditioning of mixed finite element methods (Brezzi and Fortin [1991]), Reissner-Mindlin plates (Arnold et al. [1997]) and the Brinkman equations (Vassilevski and Villa [2013]). Let A be the linear system obtained from discretization of $a(\cdot, \cdot)$ by some $\boldsymbol{H}(\mathrm{div})$-conforming finite elements of arbitrary order on a general unstructured mesh. Our goal is to design a scalable parallel solver for A.

It is well known that finding efficient iterative solvers for A is not trivial because of the "near-null space" of A. The currently available scalable parallel solvers include the auxiliary space divergence solver (ADS) (Kolev and Vassilevski [2012]) in the *hypre* library [`www.llnl.gov/CASC/hypre/`] and PCBDDC (Zampini [2016]) in the PETSc library. The former relies on the regular HX-decomposition for $\boldsymbol{H}(\mathrm{div})$ functions proposed in Hiptmair and Xu [2007]. The setup of ADS is quite involved and requires additional input from

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the user, namely, some discrete gradient and discrete curl operators. On the other hand, PCBDDC is based on the Balancing Domain Decomposition by Constraints algorithm (Dohrmann [2003]). Its construction requires that the local discrete systems are assembled at subdomain level. To accommodate high contrast and jumps in the coefficients, the primal space in PCBDDC is adaptively enriched by solving some generalized eigenvalue problems, see Zampini and Keyes [2016].

In this paper, we propose an alternative way to solve systems with A. Our approach is based on the traditional hybridization technique used in the mixed finite element method (Brezzi and Fortin [1991]), thus reducing the problem to a smaller problem for the respective Lagrange multipliers that are involved in the hybridization. The reduced problem is symmetric positive definite, and as is well-known, is H^1-equivalent. Thus, in principle, one may apply any scalable AMG solver that is suitable for H^1 problems. Unlike ADS, the hybridization approach does not require additional specialized information (such as discrete gradient and discrete curl) from the user. Instead, it requires that the original problem is given in unassembled element-based form.

One main issue that has to be addressed is the choice of the basis of the Lagrange multiplier space. In general, the reduced problem contains the constant function in its near null-space. However, if the basis for the Lagrange multipliers is not properly scaled (i.e., does not provide partition of unity), the coefficient vector of the constant functions is not a constant multiple of the vector of ones. The latter is a main assumption in the design of AMG for H^1-equivalent problems. We resolve this problem in an algebraic way by constructing a diagonal matrix which we use to rescale the reduced system such that the constant vector is the near-null space of the rescaled matrix, so that the respective AMG is correctly designed.

The proposed hybridization with diagonal rescaling is implemented in a parallel code and its scalability is tested in comparison with the state-of-the-art ADS solver. The results demonstrate that the new solver provides a competitive alternative to ADS; it outperforms ADS very clearly for higher order elements.

Although in this paper we focus on finite element problems discretized by Raviart-Thomas elements, the proposed approach can be applied to other $\boldsymbol{H}(\mathrm{div})$ conforming discretizations like Brezzi-Douglas-Marini elements, Arnold-Boffi-Falk elements (Arnold et al. [2005]), or numerically upscaled problems (Chung et al. [2015], Kalchev et al. [2016]).

The rest of the paper is organized as follows. In Sect. 2, we give a detailed description of the hybridization technique. The properties of the hybridized system are discussed in Sect. 3. After that, we present in Sect. 4 several challenging numerical examples to illustrate the performance of the method comparing it with ADS.

2 Hybridization

We consider the variational problem associated with the bilinear form (1): find $\boldsymbol{u} \in \boldsymbol{H}_0(\mathrm{div};\Omega)$ such that

$$a(\boldsymbol{u}, \boldsymbol{v}) = (\boldsymbol{f}, \boldsymbol{v}), \qquad \forall\, \boldsymbol{v} \in \boldsymbol{H}_0(\mathrm{div}; \Omega). \tag{2}$$

Here, $\boldsymbol{f}$ is a given function in $\big(L^2(\Omega)\big)^d$ and $(\cdot,\cdot)$ is the usual L^2 inner product in Ω. Our following discussion is based on discretization of the variational problem (2) by Raviart-Thomas elements of arbitrary order. We note that other $\boldsymbol{H}(\mathrm{div})$-conforming finite elements can also be considered. Let $\mathcal{T}_h$ be a general unstructured mesh on Ω. The space of Raviart-Thomas elements of order $k \geq 0$ on $\mathcal{T}_h$ will be denoted by RT_k. For instance, if $\mathcal{T}_h$ is a simplicial mesh, then RT_k is defined to be

$$RT_k = \big\{\, \boldsymbol{v}_h \in \boldsymbol{H}_0(\mathrm{div}; \Omega) \;\big|\; \boldsymbol{v}_h|_\tau \in \big(P_k(\tau)\big)^d + \boldsymbol{x} P_k(\tau) \quad \forall \tau \in \mathcal{T}_h \,\big\},$$

where $P_k(\tau)$ denotes the set of polynomials of degree at most k on τ. For definitions of RT_k on rectangular/cubic meshes, see for example Brezzi and Fortin [1991]. Discretization of (2) by RT_k elements results in a linear system of equations

$$Au = f. \tag{3}$$

We are going to formulate an equivalent problem such that the modified problem can be solved more efficiently. We note that RT_k basis functions are either associated with degrees of freedom (dofs) in the interior of elements, on boundary faces, or interior faces of a conforming finite element mesh. Those associated with dofs in the interior of elements or on boundary faces are supported in only one element, while those associated with dofs on interior faces are supported in two elements. In hybridization, the RT_k basis functions that are associated with dofs on interior faces are split into two pieces, each supported in one and only one element. In practice, the splitting can be done by making use of the element-to-dofs relation table to identify the shared dofs between any pair of neighboring elements. This relation table can be constructed during the discretization. The space of Raviart-Thomas element after the splitting will be denoted by $\widehat{RT}_k$. If we discretize $a(\cdot,\cdot)$ with the basis functions in $\widehat{RT}_k$, the resulting system will have a block diagonal matrix $\widehat{A}$. Next, we need to enforce the continuity of the split basis functions in some way such that the solution of the modified system coincides with the original problem. Suppose a RT_k basis function $\boldsymbol{\phi}$ is split into $\widehat{\boldsymbol{\phi}}_1$ and $\widehat{\boldsymbol{\phi}}_2$. The simplest way is to use Lagrange multiplier space to make the coefficient vectors of the test functions from both sides of an interior interface to be the same. If we set such constraints for all the split basis functions, we obtain a constraint matrix C.

Remark 1. There are other ways to enforce continuity of $\widehat{RT}_k$. For example, when constructing the constraint matrix C, one can also use the normal traces λ of the original RT_k basis functions as Lagrange multipliers; see Cockburn and Gopalakrishnan [2004].

The modified problem after introducing the Lagrange multipliers takes the saddle–point form

$$\begin{bmatrix} \widehat{A} & C^T \\ C & 0 \end{bmatrix} \begin{bmatrix} \widehat{u} \\ \lambda \end{bmatrix} = \begin{bmatrix} \widehat{f} \\ 0 \end{bmatrix}. \tag{4}$$

Here, $\widehat{u}$ is the coefficient vector of $\widehat{\boldsymbol{u}}_h$. The saddle point problem (4) can be reduced to

$$S\lambda = g, \tag{5}$$

where $S = C\widehat{A}^{-1}C^T$ and $g = C\widehat{A}^{-1}\widehat{f}$. The Schur complement S and the new right-hand side g can be explicitly formed very efficiently because $\widehat{A}$ is block diagonal. In fact, the inversion of $\widehat{A}$ is embarrassingly parallel. Here, each local block of $\widehat{A}$ is invertible, so $\widehat{A}^{-1}$ is well-defined. We will show in the next section that S is actually an s.p.d. system of the Lagrange multipliers, and that it can be solved efficiently by existing parallel linear solvers. After solving for λ, $\widehat{u}$ can be computed by back substitution $\widehat{u} = \widehat{A}^{-1}(\widehat{f} - C^T\lambda)$. Note that the back substitution involves only an action of $\widehat{A}^{-1}$ (already available in the computation of S) and some matrix-vector multiplications, which are inexpensive (local) and scalable computations.

3 Discussion

The hybridization approach described in the previous section can be summarized as follows:

1. Split the RT_k basis to obtain $\widehat{A}$ and $\widehat{f}$.
2. Compute $\widehat{A}^{-1}$ and form $S = C\widehat{A}^{-1}C^T$ and $g = C\widehat{A}^{-1}\widehat{f}$.
3. Solve the system $S\lambda = g$.
4. Recover $\widehat{u}$ by back substitution.

As explained in Sect. 2, step 2 and 4 are scalable (inexpensive local) computations. In contrast, step 3 involves the main computational cost. Thus, it is important that we can solve S efficiently. In this section, we describe some properties of S. First, we show that S is related to some hybridized mixed discretization of the second order differential operator $-\nabla\cdot(\beta^{-1}\nabla) + \alpha^{-1}I$ (acting on scalar functions). We note that the differential problem associated with (2) is

$$-\nabla(\alpha\nabla\cdot\boldsymbol{u}) + \beta\boldsymbol{u} = \boldsymbol{f} \tag{6}$$

with homogeneous Dirichlet boundary condition $\boldsymbol{u}\cdot\boldsymbol{n} = 0$. The latter operator acts on vector-functions. We now make the following connection between

these two operators. If we introduce an additional variable $p = \alpha \nabla \cdot \boldsymbol{u}$, then (6) becomes the following first order system (for $\boldsymbol{u}$ and p)

$$\begin{aligned} \beta \boldsymbol{u} - \nabla p &= \boldsymbol{f}, \\ \nabla \cdot \boldsymbol{u} - \alpha^{-1} p &= 0. \end{aligned} \tag{7}$$

It is noteworthy to note that the structure of (7) is the same as the mixed formulation of the differential operator $-\nabla \cdot (\beta^{-1} \nabla) + \alpha^{-1} I$. So we can apply a hybridized mixed discretization (Cockburn and Gopalakrishnan [2004, 2005]) for $-\nabla \cdot (\beta^{-1} \nabla) + \alpha^{-1} I$ to discretize (7). To apply the hybridized mixed discretization, we note that the weak form of (7) is to find $(\boldsymbol{u}, p) \in \boldsymbol{H}_0(\mathrm{div}; \Omega) \times L^2(\Omega)$ such that

$$\begin{aligned} (\beta \boldsymbol{u}, \boldsymbol{v}) + (p, \nabla \cdot \boldsymbol{v}) &= (\boldsymbol{f}, \boldsymbol{v}) && \forall\, \boldsymbol{v} \in \boldsymbol{H}_0(\mathrm{div}; \Omega) \\ (\nabla \cdot \boldsymbol{u}, q) - (\alpha^{-1} p, q) &= 0 && \forall\, q \in L^2(\Omega). \end{aligned} \tag{8}$$

Let $W_h^k \subset L^2(\Omega)$ be a space of piecewise polynomials such that RT_k and W_h^k form a stable pair for the mixed discretization of (8). For instance, for simplicial meshes, we can take

$$W_h^k = \left\{ \, q \in \boldsymbol{L}^2(\Omega) \;\middle|\; q|_\tau \in P_k(\tau) \quad \forall \tau \in \mathcal{T}_h \, \right\}.$$

If (8) is discretized by the pair $\widehat{RT}_k$-W_h^k and the continuity of $\widehat{RT}_k$ is enforced by the constraint matrix C as described in Sect. 2, we get a 3 by 3 block system of equations of the form

$$\begin{bmatrix} \widehat{M} & \widehat{B}^T & C^T \\ \widehat{B} & -W & 0 \\ C & 0 & 0 \end{bmatrix} \begin{bmatrix} \widehat{u} \\ p \\ \lambda \end{bmatrix} = \begin{bmatrix} \widehat{f} \\ 0 \\ 0 \end{bmatrix}. \tag{9}$$

As $\widehat{M}$ and W are weighted L^2 mass matrices of the spaces $\widehat{RT}_k$ and W_h^k respectively, they are invertible. Hence, the 2 by 2 block matrix $\begin{bmatrix} \widehat{M} & \widehat{B}^T \\ \widehat{B} & -W \end{bmatrix}$ is invertible, and (9) can be reduced to

$$\begin{bmatrix} C & 0 \end{bmatrix} \begin{bmatrix} \widehat{M} & \widehat{B}^T \\ \widehat{B} & -W \end{bmatrix}^{-1} \begin{bmatrix} C^T \\ 0 \end{bmatrix} \lambda = \begin{bmatrix} C & 0 \end{bmatrix} \begin{bmatrix} \widehat{M} & \widehat{B}^T \\ \widehat{B} & -W \end{bmatrix}^{-1} \begin{bmatrix} \widehat{f} \\ 0 \end{bmatrix}. \tag{10}$$

Since the $(1,1)$ block of $\begin{bmatrix} \widehat{M} & \widehat{B}^T \\ \widehat{B} & -W \end{bmatrix}^{-1}$ can be written as $(\widehat{M} + \widehat{B}^T W^{-1} \widehat{B})^{-1}$ and $\widehat{A} = \widehat{M} + \widehat{B}^T W^{-1} \widehat{B}$, the reduced problem (10) is in fact identical to (5). Therefore, the Schur complement S in (5) can be characterized by the

hybridized mixed discretization for the differential operator $-\nabla \cdot (\beta^{-1}\nabla) + \alpha^{-1}I$.

Remark 2. Actually the hybridized mixed discretization for $-\nabla \cdot (\beta^{-1}\nabla) + \alpha^{-1}I$ in Cockburn and Gopalakrishnan [2004, 2005] gives rise to the reduced system $\widetilde{S}$ for the Lagrange multiplier λ where

$$\widetilde{S} = C\Big(\widehat{M}^{-1} - \widehat{M}^{-1}\widehat{B}^T\big(\widehat{B}\widehat{M}^{-1}\widehat{B}^T + W\big)^{-1}\widehat{B}\widehat{M}^{-1}\Big)C^T.$$

However, since W is invertible, an application of the Sherman-Morrison-Woodbury formula implies that $\widetilde{S} = S$.

In Cockburn and Gopalakrishnan [2005], the authors proved that S is spectrally equivalent to the norm $|||\cdot|||$ on the space of Lagrange multipliers defined as

$$|||\lambda|||^2 = \sum_{\tau \in \mathcal{T}_h} \frac{1}{|\partial\tau|}\|\lambda - m_\tau(\lambda)\|^2_{\partial\tau}$$

where $m_\tau(\lambda) = \frac{1}{|\partial\tau|}\int_{\partial\tau} \lambda \, ds$. More precisely, there are constants C_1 and C_2, depending only on the approximation order k, the coefficients α, β of the operator, and the shape regularity of $\mathcal{T}_h$ such that $C_1|||\lambda|||^2 \leq \lambda^T S\lambda \leq C_2|||\lambda|||^2$ for all λ. Consequently, S is symmetric positive definite. Moreover, this shows that the near-null space of S is spanned by the constant functions, which is the main assumption to successfully apply solvers of AMG type. When solving with S, we opt for the parallel algebraic multigrid solver BoomerAMG (Henson and Yang [2002]) from the *hypre* library.

Depending on the choice of basis for the Lagrange multipliers space, the coefficient vector of a constant function is not necessarily a constant vector and the latter affects adversely the performance of classical AMG methods such as BoomerAMG from hypre. To resolve this issue, we chose to rescale S by a diagonal matrix D such that the constant vector is now in the near-null space of D^TSD. To achieve this, we solve the homogeneous problem $Sd = 0$ by applying a few smoothing steps to a random initial guess. In our numerical experiments to be presented in the next section, we use 5 conjugate gradient (CG) iterations preconditioned by the Jacobi smoother in the computation of d, which is fairly inexpensive. Once d is computed, we set $D_{ii} = d_i$ (the i-th entry of d). Noticing that $D\mathbf{1} = d$, so $\mathbf{1}$ is in the near-null space of D^TSD. We can then apply CG preconditioned by BoomerAMG constructed from D^TSD to efficiently solve the system

$$(D^TSD)\lambda_D = D^Tg.$$

Lastly, the original Lagrange multiplier λ is recovered simply by setting $\lambda = D\lambda_D$.

Another useful feature of S is that its size is less than or equal to the size of the original matrix A. This is because there is a one-to-one correspondence between Lagrange multipliers and Raviart-Thomas basis functions associated

with interior faces. For higher order Raviart-Thomas elements, a portion of the basis functions are associated with interior of elements. These basis functions are supported in one element only, so they do not need Lagrange multipliers to enforce their continuity. Hence, for higher order approximations, methods for solving with S are likely to be more efficient and faster than solving with A (using state-of-the-art solvers such as ADS) which is confirmed by our experiments.

4 Numerical Examples

In this section, we present some numerical results regarding the performance of our hybridization AMG solver. The numerical results are generated using MFEM [`mfem.org`], a scalable C++ library for finite element methods developed in the Lawrence Livermore National Laboratory (LLNL). All the experiments are performed on the cluster Sierra at LLNL. Sierra has a total of 1944 nodes (Intel Xeon EP X5660 clocked at 2.80 GHz), which are connected by InfiniBand QDR. Each node has 12 cores and equipped with 24 GB of memory.

In the solution process, the hybridized system with S is rescaled by the diagonal matrix D as described in the previous section. The rescaled system D^TSD is then solved by the CG method preconditioned with BoomerAMG (constructed from D^TSD) from the *hypre* library. As one of our goals is to compare the hybridization AMG solver with ADS, we present also the performance of ADS in all the examples. In order to have fair comparisons, the time to solution for the hybridization AMG solver includes the formation time of the Schur complement S, the computation time to construct the rescaling matrix D, the solve time for the problem with the modified matrix D^TSD by CG preconditioned by BoomerAMG, and the recovery time of the original unknown $\boldsymbol{u}$. The time to solution for ADS is simply the solve time for the original problem with A by the CG preconditioned by ADS. For the tables in the present section, #proc refers to the number of processors, while #iter refers to the number of PCG iterations.

4.1 Weak Scaling

We first test the weak scaling of the hybridization AMG solver. The problem setting is as follows. We solve problem (3) obtained by RT_k discretization on uniform tetrahedral mesh in 3D. Starting from some initial tetrahedral mesh, we refine the mesh uniformly. The problem size increases by about 8 times after one such refinement. At the same time, the number of processors for solving the refined problem is increased 8 times so that the problem size per

processor is kept roughly the same. Both the lowest order Raviart-Thomas

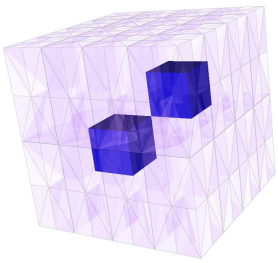


Fig. 1 Initial mesh for the RT_2 weak scaling test case. Blue region indicates Ω_i

elements RT_0 and a higher order elements, RT_2, are considered. We solve a heterogeneous coefficient problem on the unit cube, i.e. $\Omega = [0,1]^3$. The boundary conditions are $\boldsymbol{u} \cdot \boldsymbol{n} = 0$ on $\partial\Omega$, and the source function $\boldsymbol{f}$ is the constant vector $[1,1,1]^T$. Let $\Omega_i = [\frac{1}{4}, \frac{1}{2}]^3 \cup [\frac{1}{2}, \frac{3}{4}]^3$. We consider β being constant 1 throughout the domain, whereas $\alpha = \begin{cases} 1 & \text{in } \Omega \backslash \Omega_i \\ 10^p & \text{in } \Omega_i \end{cases}$ and we choose p = -4, 0, or 4. For RT_2 test case, we first partition Ω into 8 x 8 x 4 parallelepipeds. The initial tetrahedral mesh in this case is then obtained by subdividing each parallelepiped into tetrahedrons, see Fig. 1. The initial mesh of the RT_0 test case is obtained by refining the initial mesh of the RT_2 test case 3 times. The PCG iterations are stopped when the l_2 norm of the residual is reduced by a factor of 10^{10}. The time to solution (in seconds) of both the hybridization AMG and ADS for the RT_0 case are shown in Table 1. Additionally, we also report the number of PCG iterations in the brackets. We see that the number of iterations of the hybridization solver are very

Table 1 Time to solution (in seconds) in the weak scaling test: RT_0 on tetrahedral meshes, the corresponding number of PCG iterations are the reported in the brackets

#proc	Problem size	p = -4	$p = 0$	$p = 4$
Hybridization-BoomerAMG-CG				
3	200,704	0.97 (24)	0.96 (21)	0.93 (21)
24	1,589,248	1.15 (24)	1.15 (23)	1.16 (23)
192	12,648,448	1.45 (27)	1.48 (25)	1.43 (24)
1,536	100,925,440	3.31 (29)	3.03 (28)	3.03 (28)
ADS-CG				
3	200,704	2.68 (21)	1.74 (10)	1.79 (11)
24	1,589,248	4.04 (25)	3.53 (13)	3.54 (13)
192	12,648,448	7.10 (27)	5.73 (15)	5.61 (14)
1,536	100,925,440	8.30 (28)	6.28 (15)	6.51 (15)

stable against problem size and the heterogeneity of α. The average time to

solution of the hybridization approach is about 2 times faster than that of ADS. The solution time difference between the two solvers is more significant in the high order discretization case. This is due to the fact that size of the hybridized system S is much smaller than the size of the original system A. Indeed, in the case of RT_2, the average time to solution of the hybridization approach is about 8 times faster than that of ADS, see Table 2. In Fig. 2, we plot the solution time of both solvers where $p = 4$ in the definition of α. We can see that the hybridization solver has promising weak scaling over a range of nearly three decades.

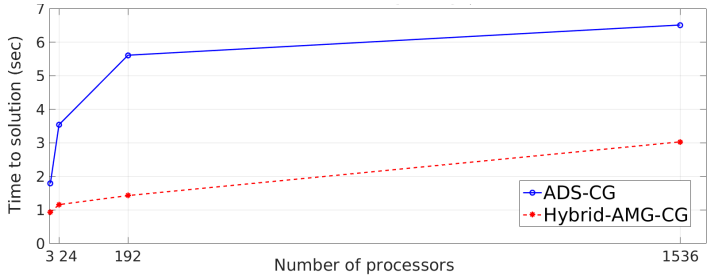
Table 2 Time to solution (in seconds) in the weak scaling test: RT_2 on tetrahedral meshes, the corresponding number of PCG iterations are the reported in the brackets

#proc	Problem size	p = -4	$p = 0$	$p = 4$
Hybridization-BoomerAMG-CG				
3	38,400	0.30 (15)	0.31 (16)	0.31 (16)
24	301,056	0.48 (18)	0.50 (21)	0.48 (20)
192	2,383,872	0.75 (28)	0.89 (29)	0.77 (29)
1,536	18,972,672	1.97 (44)	1.95 (47)	2.10 (47)
ADS-CG				
3	38,400	4.85 (23)	3.55 (13)	3.80 (14)
24	301,056	7.24 (29)	5.47 (18)	5.73 (20)
192	2,383,872	11.56 (37)	8.89 (25)	9.56 (28)
1,536	18,972,672	24.28 (53)	16.51 (37)	16.37 (39)

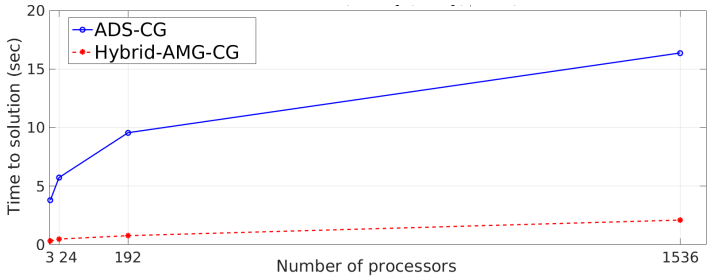
4.2 Strong Scaling

In the second example, we investigate the strong scaling of the hybridization AMG solver. The problem considered in this section is the crooked pipe problem, see Kolev and Vassilevski [2012] for a detailed description of the problem. The mesh for this problem is depicted in Fig. 3. The coefficient α and β are piecewise constants. More precisely, $(\alpha, \beta) = (1.641, 0.2)$ in the red region, and $(\alpha, \beta) = (0.00188, 2000)$ in the blue region. The difficulties of this problem are the large jumps of coefficients and the highly stretched elements in the mesh (see Fig. 3). For this test, the problem is discretized by RT_1. The size of A is 2,805,520, and we solve the problem using 4, 8, 16 ,32 and 64 processors. The PCG iterations are stopped when the l_2 norm of the residual is reduced by a factor of 10^{14}. The number of PCG iterations and time to solution are reported in Table 3, and we plot the speedup in Fig. 4. When measuring the speedup, solution times are corrected by the number of iterations.

Both solvers exhibit good strong scaling. We note that in this example, the solution time of the hybridization AMG solver is much smaller than the ADS



(a) RT_0



(b) RT_2

Fig. 2 Weak scaling comparisons between the hybridization AMG solver (red dotted line) and ADS (blue solid line)

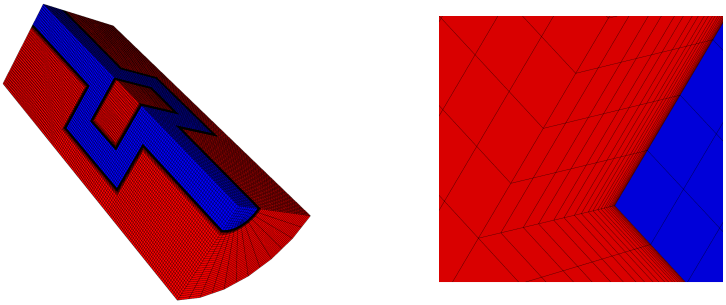


Fig. 3 The mesh for the Crooked Pipe problem (left). A dense layer of highly stretched elements (right) has been added to the neighborhood of the material interface in the exterior subdomain in order to resolve the physical diffusion

solver. The average solve time of the hybridization AMG solver is about 10 times smaller than that of ADS. In particular, the hybridization AMG solver with 4 processors is still 2 times faster than ADS with 64 processors. The difference in the computation time for this example is highly noticeable.

Lastly, we report the time spent on different components of the hybridization approach in Table 4. We observe that except for solving with S (i.e. setup and PCG solve), the other components scale fairly well. Also, as we point out

Table 3 Strong scaling test, original problem size: 2,805,520

	Hybridization-BoomerAMG-CG		ADS-CG	
#proc	#iter	Time to solution	#iter	Time to solution
4	25	23.46	32	508.66
8	31	14.21	32	251.37
16	28	6.83	33	130.26
32	28	3.98	34	73.47
64	31	2.92	34	54.58

in Sect. 3, solving with S is the most time consuming part of the hybridization AMG code. We remark that during the formation of S, we stored the inverses of local blocks of $\widehat{A}$. So when we recover u by back substitution, only matrix multiplication is needed. Hence, the recovery of u is extremely cheap and scalable.

Table 4 Timing of each component of the new solver

#proc	Formation of S	Computation of D	Setup	PCG solve	Recovery of u
4	7.55	0.22	3.87	11.72	0.092
8	3.95	0.11	2.29	7.81	0.046
16	1.84	0.057	1.4	3.52	0.022
32	1.11	0.034	0.83	2.01	0.012
64	0.68	0.027	0.52	1.7	0.006

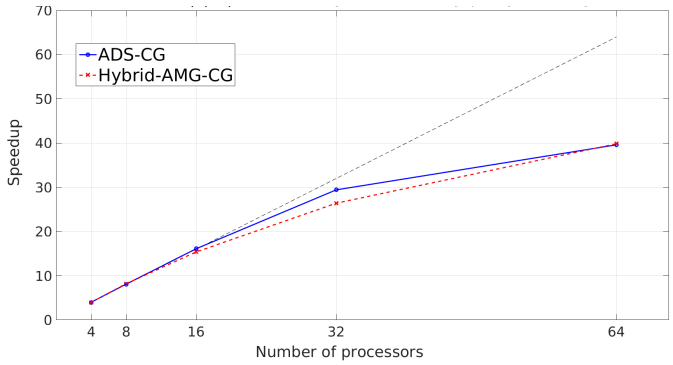


Fig. 4 Strong scaling comparison between the hybridization AMG solver (red dotted line) and ADS (blue solid line). Black dotted line indicates perfect scaling

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