# Dual Schur method in time for nonlinear ODE

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## 1 Introduction

We developed parallel time domain decomposition methods to solve systems of linear ordinary differential equations (ODEs) based on the Aitken-Schwarz [5] or primal Schur complement domain decomposition methods [4]. The methods require the transformation of the initial value problem in time defined on $]0,T]$ into a time boundary values problem. Let $f(t,y(t))$ be a function belonging to $\mathscr{C}^1(\mathbb{R}^+,\mathbb{R}^d)$ and consider the Cauchy problem for the first order ODE:

$$\begin{cases} \dot{y} = f(t,y(t)),\, t \in ]0,T],\ y(0) = \alpha \in \mathbb{R}^d. \end{cases} \tag{1}$$

The time interval $[0,T]$ is split into $p$ time slices $S^{(i)} = [T^+_{i-1}, T^-_i]$, with $T^+_0 = 0$ and $T^-_p = T^-$. The difficulty is to match the solutions $y_i(t)$ defined on $S^{(i)}$ at the boundaries $T^+_{i-1}$ and $T^-_i$. Most of time domain decomposition methods are shooting methods [1] where the jumps $y_i(T^-_i) - y_{i+1}(T^+_i)$ are corrected by a sequential process which is propagated in the forward direction (i.e. the correction on the time slice $S^{(i-1)}$ is needed to compute the correction on time slice $S^{(i)}$). Our approach consists in breaking the sequentiality of the update of each time slice initial value. To this end, we transform the initial value problem (IVP) into a boundary values problem (BVP) leading to a second order ODE:

$$\begin{cases} \ddot{y}(t) = g(t,y(t),\dot{y}(t)) \overset{def}{=} \dfrac{\partial f}{\partial t}(t,y(t)) + \dot{y}(t)\dfrac{\partial f}{\partial y}(t,y(t)),\, t \in ]0,T[, & \text{(2a)} \\ y(0) = \alpha, & \text{(2b)} \\ \dot{y}(T) = \beta \overset{def}{=} f(T,y(T)) & \text{(2c)} \end{cases}$$

Then classical domain decomposition methods apply such as the multiplicative Schwarz method with no overlapping time slices and Dirichlet-Neumann transmission conditions (T.C.) for linear system of ODE (or PDE [6]). As proved in [5] the convergence/divergence of the error at the boundaries of this Schwarz time DDM can be accelerated by the Aitken technique to the right solution when $f(t,y(t))$ is linear. Nevertheless, the difficulty in solving equation (2) is that $\beta$ is not given by the original IVP. In [7] when $f(t,y(t))$ is nonlinear with respect of $y(t)$ and scalar, we

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proposed to replace the end boundary condition (2c) by imposing, if $f(T,y(T)) \neq 0$, the invariant flux condition for $t = T$:

$$(f(T,y(T))^{-1}\dot{y}(T) = 1. \quad (3a)$$

We also showed that the right T.C. between time slices must involve the nonlinear flux condition $(f(T_i^-,y(T_i^-))^{-1}\dot{y}_i(T_i^-) = (f(T_{i+1}^+,y(T_{i+1}^+))^{-1}\dot{y}_{i+1}(T_{i+1}^+)$. In this case, we showed that the behavior of the Schwarz method with an appropriate nonlinear change of variable $\Theta$ is linear. Then, it is possible to apply the Aitken acceleration by using $\Theta$ if it is known. To overcome the lack of knowledge of $\Theta$, we propose in this paper to replace the Schwarz method by a Schur complement method.

In section 2, we recall some results on the existence and uniqueness of the proposed BVP. Section 3 gives the dual Schur complement method intimely related to the Newton step solving. The choice of T.C. to define the time slice function is discussed there. Some numerical results are given in section 4 before the conclusion.

## 2 Existence and uniqueness of the BVP solution

The problem (2) with $d = 1$ is a particular case of the more general problem:

$$\begin{cases} \ddot{y} = g(t,y,\dot{y}), a \leq t \leq b, & (4a) \\ a_0\, y(a) - a_1\, \dot{y}(a) = \alpha, |a_0| + |a_1| \neq 0, & (4b) \\ b_0\, y(b) + b_1\, \dot{y}(b) = \beta, |b_0| + |b_1| \neq 0. & (4c) \end{cases}$$

H.B. Keller [3] has established the existence and uniqueness of a solution to problem (4) under the hyppotheses of monotonicity and upper bound on the partial derivatives of $g$ in the theorem that follows:

**Theorem 1 (H.B. Keller).** *Let $g(t,y,\dot{y})$ have continuous derivatives which satisfy:*

$$\frac{\partial g(t,y(t),\dot{y}(t))}{\partial y} > 0, \left|\frac{\partial g(t,y(t),\dot{y}(t))}{\partial \dot{y}}\right| \leq M, \quad (5)$$

*for some $M \geq 0$, $a \leq t \leq b$ and all continuously differentiable functions $y(t)$. Let the constants $a_i, b_i$ satisfy:*

$$a_i \geq 0, b_i \geq 0, i = 0,1; a_0 + b_0 > 0. \quad (6)$$

*then a unique solution of* (4a), (4b), (4c) *exists for each* $(\alpha,\beta)$.

## 3 Dual Schur complement time DDM

### 3.1 BVP discretizing and it solution

Problem (2a), (2b), (3a), is discretized using a Störmer-Verlet implicit scheme [2] with $N_g+1$ regular time steps with $\Delta t = T/N_g$ over the time interval $[0,T]$. Solving it requires to find the zero of the function $F(u): \mathbb{R}^{N_g} \to \mathbb{R}^{N_g}$ with $u_j \simeq u(t_j)$, $t_j = (j-1)\Delta t$ and defined as:

$$F(u) = \begin{pmatrix} u_0 - \alpha \\ u_{j+1} - 2u_j + u_{j-1} - \Delta t^2 g(t_j, u_j),\, j = 1, \ldots, N_g - 1 \\ f^{-1}(t_{N_g}, u_{N_g}) B(u_{N_g}) - 1 \end{pmatrix} \tag{7}$$

where $g(t,u) \stackrel{def}{=} \frac{\partial f}{\partial t}(t,u) + f(t,u)\frac{\partial f}{\partial u}(t,u)$, and $B(u_{N_g})$ corresponds to the discretizing of $\dot{u}(T)$ as:

$$B(u_{N_g}) = \frac{3u_{N_g} - 4u_{N_g-1} + u_{N_g-2}}{2\Delta t} \simeq \dot{u}(T) + O(\Delta t^2) \tag{8a}$$

$$B(u_{N_g}) = \frac{11u_{N_g} - 18u_{N_g-1} + 9u_{N_g-2} - 2u_{N_g-3}}{6\Delta t} \simeq \dot{u}(T) + O(\Delta t^3) \tag{8b}$$

We applied the Newton method to find the zero of function $F(u)$. Starting from an initial guess, it writes if $||F(u^m)|| > \varepsilon$ for the $(m+1)$-th iteration:

$$h^m = -(\nabla_u F(u^m))^{-1} F(u^m),\; u^{m+1} = u^m + h^m. \tag{9}$$

Let us notice that the Newton method is sensitive to the initial solution. One can consider to search the initial solution by performing a few Newton iterations on different coarse levels of time grid discretizing. The approximate solution obtained on a previous coarse grid gives the initial guess solution for the next time grid after interpolating. There is no Courant-Friedrich-Lax stability condition because we use an implicit Störmer-Verlet scheme.

### 3.2 Dual Schur complement in time formulation

For the time domain decomposition, we split the time intervalle $[0,T]$ in $p$ slices $S^{(i)}, i = 1, \ldots, p$ and we denote by $u^{(i)}$ the solution on the $i$-th time slice $S^{(i)}$. For the sake of simplicity and without loss of generality we set all the time slices to have the same size and use $N+1$ regular time steps on each such that $S^{(i)} = [t_0^{(i)}, t_N^{(i)}] \stackrel{def}{=} [(i-1)N\Delta t, iN\Delta t]$ (then the total number of time steps on $[0,T]$ is $N_g + 1 = p \times N + 1$). Here, the main idea consists in finding the zero of the local function $F_i$ defined on the time slice $S^{(i)}$ under the continuity constraint of the solution at the time slices boundaries. Two strategies can be applied to define the transmission conditions (T.C.) of the local function $F_i(u^{(i)})$:

1. The first strategy S1 considers the original function $F$ and split its components at the time slices boundaries in two parts. Each one corresponds to the contribution of the solution components belonging to the time slice under consideration (10b) at $j=0$ for $S^{(i)}, i=2,\dots,p$ and (12b) at $j=N$ for $S^{(i)}, i=1,\dots,p-1$.
2. The second strategy S2 considers the T.C. corresponding to the nonlinear flux (10c) at $j=0$ for $S^{(i)}, i=2,\dots,p$ and (12c) at $j=N$ for $S^{(i)}, i=1,\dots,p-1$.

$$\begin{cases} (F_1(u))_0 = u_0 - \alpha & (10a) \\ S1: (F_i(u))_0 = u_1 - u_0 - \dfrac{1}{2}\Delta t^2 g(t_0^{(i)}, u_0), i=2,\dots,p & (10b) \\ S2: (F_i(u))_0 = f^{-1}(t_0^{(i)}, u_0)B(u_0), i=2,\dots,p & (10c) \end{cases}$$

$$\begin{cases} (F_i(u))_j = u_{j+1} - 2u_j + u_{j-1} - \Delta t^2 g(t_j^{(i)}, u_j), & (11a) \\ j=1,\dots,N-1,\ i=1,\dots,p \end{cases}$$

$$\begin{cases} (F_p(u))_N = f^{-1}(t_N^{(p)}, u)B(u_N) - 1 & (12a) \\ S1: (F_i(u))_N = -u_N + u_{N-1} - \dfrac{1}{2}\Delta t^2 g(t_N^{(i)}, u_N),\ i=2,\dots,p-1 & (12b) \\ S2: (F_i(u))_N = f^{-1}(t_N^{(i)}, u_N)B(u_N),\ i=2,\dots,p-1 & (12c) \end{cases}$$

Then, we use the Newton method on each time slices $S^{(i)}$ and introduce the Lagrange multipliers $\lambda_i, i=1,\dots,p-1$ to ensure the continuity of the solution between the time slices (adding this Lagrange multiplier to (10b) (respectively (10c)) and substracting it to (12b) (respectively (12c))). It writes:

$$h^{(i),m} = u^{(i),m+1} - u^{(i),m} = -(\nabla F_i(u^{(i),m}))^{-1}(F(u^{(i),m}) + \underbrace{(\lambda_{i-1}, 0, \dots, 0, -\lambda_i)^t}_{\in \mathbb{R}^{N+1}}) \quad (13)$$

with the constraints

$$u_0^{(i),m} + h_0^{(i),m} = u_N^{(i-1),m} + h_N^{(i-1),m}, i=2,...,p \quad (14)$$

Let us give the computing details. Introducing the Jacobian matrix $J^{(i)}$ corresponding to $\nabla F_i(u^{(i),m})$, the index $I$ for the unknowns $[1,\dots,N-1]$ and $E$ for the unknowns $0,N$, the linearized system of the Newton step writes after a permutation of unknowns:

$$\begin{pmatrix} J_{II}^{(i)} & J_{I\Gamma}^{(i)} \\ J_{\Gamma I}^{(i)} & J_{\Gamma\Gamma}^{(i)} \end{pmatrix} \begin{pmatrix} h_I^{(i)} \\ h_E^{(i)} \end{pmatrix} = \begin{pmatrix} b_I^{(i)} \\ b_E^{(i)} \end{pmatrix} + \begin{pmatrix} 0 \\ \Lambda_i \end{pmatrix} \quad (15)$$

where

$$(\Lambda_i, b_I^{(i)}, b_E^{(i)}) = \begin{cases} (-\lambda_1, -(F_1(u^{(1),m})_{0,\ldots,N-1}, -(F_1(u^{(1),m})_N) & i=1 \\ (\begin{pmatrix} \lambda_{i-1} \\ -\lambda_i \end{pmatrix}, -(F_i(u^{(i),m})_{1,\ldots,N-1}, -(F_i(u^{(i),m})_{[0,N]}) & i \neq \{1,p\} \\ \lambda_{p-1}, -(F_p(u^{(p),m})_{1,\ldots,N}, -(F_p(u^{(p),m})_0) & i=p \end{cases}$$

if $h_E^{(i)}$ is known then the first line of system (15) gives:

$$h_I^{(i)} = (J_{II}^{(i)})^{-1}(b_E^{(i)} - J_{I\Gamma}^{(i)} h_E^{(i)}) \tag{16}$$

Reporting $h_I^{(i)}$ in the second line of system (15), we obtain:

$$S_\Gamma^{(i)} h_E^{(i)} \stackrel{def}{=} (J_{\Gamma\Gamma}^{(i)} - J_{\Gamma I}^{(i)} (J_{II}^{(i)})^{-1} J_{I\Gamma}^{(i)}) h_E^{(i)} = (b_E^{(i)} - (J_{II}^{(i)})^{-1} b_I^{(i)}) + \Lambda_i \tag{17}$$

If $\Lambda_i$ is known then $h_E^{(i)}$ can be computed. To compute $\Lambda_i$, we impose the continuity of the solution among the time slices:

$$\begin{pmatrix} u_0^{(i)} + h_0^{(i)} \\ u_N^{(i)} + h_N^{(i)} \end{pmatrix} = \begin{pmatrix} u_N^{(i-1)} + h_N^{(i-1)} \\ u_0^{(i+1)} + h_0^{(i+1)} \end{pmatrix} \tag{18}$$

$$\begin{pmatrix} h_0^{(i)} \\ h_N^{(i)} \end{pmatrix} = \begin{pmatrix} \bar{S}_{\Gamma,00}^{(i)} & \bar{S}_{\Gamma,0N}^{(i)} \\ \bar{S}_{\Gamma,N0}^{(i)} & \bar{S}_{\Gamma,NN}^{(i)} \end{pmatrix} \begin{pmatrix} g_0^{(i)} + \lambda_{i-1} \\ g_N^{(i)} - \lambda_i \end{pmatrix} \stackrel{def}{=} \bar{S}_\Gamma^{(i)} \begin{pmatrix} g_0^{(i)} + \lambda_{i-1} \\ g_N^{(i)} - \lambda_i \end{pmatrix} \tag{19}$$

where

$$\bar{S}_{\Gamma,N}^{(1)} \stackrel{def}{=} (S_\Gamma^{(1)})^{-1}, \begin{pmatrix} \bar{S}_{\Gamma,00}^{(i)} & \bar{S}_{\Gamma,0N}^{(i)} \\ \bar{S}_{\Gamma,N0}^{(i)} & \bar{S}_{\Gamma,NN}^{(i)} \end{pmatrix} \stackrel{def}{=} (S_\Gamma^{(i)})^{-1}, i=2,\ldots,p-1, \bar{S}_{\Gamma,0}^{(p)} \stackrel{def}{=} (S_\Gamma^{(p)})^{-1}.$$

We obtain the Lagrange multipliers tridiagonal system (20) of the form $M(\lambda_1,\ldots,\lambda_{p-1})^t = (b_\Gamma^{(1)},\ldots,b_\Gamma^{(p-1)})^t$ that links all the time slices and allows the instantaneous propagation of the information between all the time slices:

$$\begin{cases} & -(\bar{S}_{\Gamma,N}^{(1)} + \bar{S}_{\Gamma,00}^{(2)})\lambda_1 & +\bar{S}_{\Gamma,0N}^{(2)}\lambda_2 & = b_\Gamma^{(1)} \\ \bar{S}_{\Gamma,N0}^{(i-1)}\lambda_{i-2} & -(\bar{S}_{\Gamma,NN}^{(i-1)} + \bar{S}_{\Gamma,00}^{(i)})\lambda_{i-1} & +\bar{S}_{\Gamma,0N}^{(i)}\lambda_i & = b_\Gamma^{(i-1)}, i=3,\ldots,p-1 \\ \bar{S}_{\Gamma,N0}^{(p-1)}\lambda_{p-2} & -(\bar{S}_{\Gamma,NN}^{(p-1)} + \bar{S}_{\Gamma,0}^{(p)})\lambda_{p-1} & & = b_\Gamma^{(p-1)} \end{cases} \tag{20}$$

with

$$\begin{aligned} b_\Gamma^{(1)} &= u_0^{(2)} - u_N^{(1)} - S_{\Gamma,N}^{(1)} g_N^{(1)} + S_{\Gamma,00}^{(2)} g_0^{(2)} + S_{\Gamma,0N}^{(2)} g_N^{(2)} \\ b_\Gamma^{(i-1)} &= u_0^{(i)} - u_N^{(i-1)} - \bar{S}_{\Gamma,N0}^{(i-1)} g_0^{(i-1)} - \bar{S}_{\Gamma,NN}^{(i-1)} g_N^{(i-1)} + \bar{S}_{\Gamma,00}^{(i)} g_0^{(i)} + \bar{S}_{\Gamma,0N}^{(i)} g_N^{(i)}, i=3,\ldots,p-1, \\ b_\Gamma^{(p-1)} &= u_0^{(p)} - u_N^{(p-1)} - \bar{S}_{\Gamma,N0}^{(p-1)} g_0^{(p-1)} - \bar{S}_{\Gamma,NN}^{(p-1)} g_N^{(p-1)} + \bar{S}_{\Gamma,0}^{(p)} g_0^{(p)} \end{aligned}$$

## 4 Numerical results of the Schur time DDM

We tested our Schur time DDM on the IVP (1) with $f(t,y) = 1 + y^3(t)$ leading to $g(t,y,\dot{y}) = \dot{y}(t)(3y^2(t))$. The number of time steps is $N_g = 2000$ over $[0,1]$ and $\alpha = 1$.The monotonicity hypothesis of theorem 1 is satisfied because $y(t)$ is an increasing function on $[0,1]$ and $\alpha > 0$. The upper bound hypothesis is satisfied on interval $[0,b]$ for $b$ taken sufficiently small, because $f(t,y)$ is continuous in $y$. The initial guess is computed using 2 Newton iterations on each of the two coarse grids of 20 and 200 time steps respectively. The initial $||F||_2$ is then around $10^{-2}$. Let us notice that Newton's method on the coarsest time mesh does not converge to the solution of the problem. Table 1 shows that both strategies for T.C. (10b)

| $N_g = 2000$ | T.C. : (10b) (12b), **B(u) :** (8b) | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $p$ | 1 | 2 | 4 | 8 | 10 | 25 | 50 | 100 |
| #it | 5 | 5 | 5 | 5 | 5(6) | 5 (6) | 5 | 5 |
| $log_{10}(\|\|F\|\|_2)$ | -13.02 | -7.56 | -7.55 | -7.55 | -7.49 | -7.59 | -7.52 | -7.55 |
| $log_{10}(\|\|h\|\|_2)$ | -5.42 | -6.33 | -6.31 | -6.28 | -5.41 | -5.94 | -6.19 | -6.18 |
| $log_{10}(min(\kappa_2(\bar{S}_\Gamma^{(i)})))$ | - | - | 0.96 | 1.15 | 1.24 | 1.75 | 4.00 | 2.78 |
| $log_{10}(max(\kappa_2(\bar{S}_\Gamma^{(i)})))$ | - | - | 1.40 | 2.35 | 2.65 | 3.85 | 5.65 | 5.65 |
| $log_{10}(min(\kappa_2(M)))$ | - | 0 | 1.31 | 2.75 | 3.14 | 4.58 | 3.99 | 6.50 |
| $log_{10}(max(\kappa_2(M))$ | - | 0 | 1.64 | 3.05 | 3.45 | 4.89 | 4.00 | 6.82 |

| $N_g = 2000$ | T.C. : (10c) (12c), B(u) : (8a) | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $p$ | 1 | 2 | 4 | 8 | 10 | 25 | 50 | 100 |
| #it | 5 | 5 | 5 | 5 | 5 | 5 | 9 | - (8) |
| $log_{10}(\|\|F\|\|_2)$ | -12.66 | -10.62 | -10.62 | -9.73 | -10.00 | -8.37 | -7.12 | -6.04 |
| $log_{10}(\|\|h\|\|_2)$ | -7.29 | -7.30 | -7.11 | -7.05 | -6.97 | -6.58 | 6.06 | -5.47 |
| $log_{10}(min(\kappa_2(\bar{S}_\Gamma^{(i)})))$ | - | - | 3.05 | 3.25 | 3.30 | 3.62 | 5.97 | 4.52 |
| $log_{10}(max(\kappa_2(\bar{S}_\Gamma^{(i)})))$ | - | - | 5.70 | 6.23 | 7.31 | 8.29 | 10.68 | 10.68 |
| $log_{10}(min(\kappa_2(M)))$ | - | 0 | 1.97 | 2.82 | 3.12 | 4.31 | 5.95 | 6.03 |
| $log_{10}(max(\kappa_2(M))$ | - | 0 | 3.38 | 3.87 | 4.73 | 5.78 | 7.19 | 8.45 |

| $N_g = 2000$ | T.C. : (10c) (12c), **B(u) :** (8b) | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $p$ | 1 | 2 | 4 | 8 | 10 | 25 | 50 | 100 |
| #it | 5 | 6 | 6 | 6 | 6 | 15 | - | - |
| $log_{10}(\|\|F\|\|_2)$ | -13.02 | -10.96 | -9.67 | -8.31 | -8.60 | -7.48 | -6.76 | -5.59 |
| $log_{10}(\|\|h\|\|_2)$ | -5.42 | -8.11 | -7.83 | -7.52 | -7.87 | -5.97 | -4.11 | -3.56 |
| $log_{10}(min(\kappa_2(\bar{S}_\Gamma^{(i)})))$ | - | - | 2.75 | 3.12 | 3.21 | 3.62 | 5.55 | 4.54 |
| $log_{10}(max(\kappa_2(\bar{S}_\Gamma^{(i)})))$ | - | - | 6.35 | 6.52 | 7.08 | 9.05 | 11.08 | 11.08 |
| $log_{10}(min(\kappa_2(M)))$ | - | 0 | 1.73 | 2.47 | 3.24 | 4.45 | 5.59 | 6.23 |
| $log_{10}(max(\kappa_2(M))$ | - | 0 | 3.34 | 3.39 | 4.23 | 6.09 | 7.85 | 8.85 |

**Table 1** Number of Newton iterations #it, with respect to the number of time slices $p$, required to reach $log_{10}(||f||_2) < -7$ and $log_{10}(||f||_2) < -6$ and with the two discretizing of $B(u)$. $log_{10}(min/max(\kappa_2(\bar{S}_\Gamma^{(i)})))$ (respectively $log_{10}(min/max(\kappa_2(M)))$ refers to the minimum or maximum value of the condition number of the local Schur complement for the time slices 2 to $p-1$ (respectively of the Lagrange multipliers system) over the Newton iterations.

(12b) or (10c) (12c) work well until the number ten of time slices. The first strategy seems to be more robust until $p = 100$ time slices. For $p = 50$ and $p = 100$ time slices the method does not reach the convergence criterion and oscillates with $||F||_2$ around $10^{-5}$. These oscillations are mainly due to the local Schur complement of the time slices 2 to $p-1$ where its condition number maximum value, over all the Newton iterations, reaches around $10^{11}$ for some time slices. Even with this local bad condition numbers, the condition number for the Lagrange multipliers system is around $10^9$ (symbol $-$ in row #it means no convergence and (8) means the iteration number among 21 iterations where the minimum values of $||F||_2$ and $||h||_2$ have been reached).

Nevertheless the right T.C. are (10c) (12c) as shown in [7] and illustrated by the following results for $f(t,y) = \sqrt{y(t)+2}$ on $[0,3]$ with $\alpha = 0.5$. The initial guess is computed with 2 (respectively 1) Newton iterations on the coarsest (respectively intermediate) time grid leading to $||F||_2 \simeq 10^{-4}$. Table 2 shows that T.C. (10b)(12b)

| $N_g = 2000$ | T.C. : (10b) (12b), B(u) : (8a) | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $p$ | 1 | 2 | 4 | 8 | 10 | 25 | 50 | 100 |
| #it | 3 | 3 | - | - | - | - | - | - |
| $log_{10}(\|\|F\|\|_2)$ | -13.02 | -10.51 | - | - | - | - | - | - |
| $log_{10}(\|\|h\|\|_2)$ | -7.43 | -7.85 | - | - | - | - | - | - |

| $N_g = 2000$ | T.C. : (10c) (12c), B(u) : (8a) | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $p$ | 1 | 2 | 4 | 8 | 10 | 25 | 50 | 100 |
| #it | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 10 |
| $log_{10}(\|\|F\|\|_2)$ | -12.84 | -10.58 | -9.89 | -8.96 | -8.26 | -6.00 | -6.06 | -6.06 |
| $log_{10}(\|\|h\|\|_2)$ | -7.43 | -8.46 | -7.05 | -7.22 | -6.10 | -5.37 | -5.27 | -5.60 |
| $log_{10}(min(\kappa_2(\bar{S}_\Gamma^{(i)})))$ | - | - | 4.98 | 5.25 | 5.25 | 5.28 | 6.33 | 5.90 |
| $log_{10}(max(\kappa_2(\bar{S}_\Gamma^{(i)})))$ | - | - | 5.88 | 6.58 | 7.55 | 8.70 | 8.66 | 8.66 |
| $log_{10}(min(\kappa_2(M)))$ | - | 0 | 1.67 | 2.25 | 2.55 | 4.49 | 6.36 | 4.65 |
| $log_{10}(max(\kappa_2(M)))$ | - | 0 | 2.27 | 2.84 | 3.84 | 5.19 | 7.96 | 5.91 |

**Table 2** Number of Newton iterations #it, with respect to the number of time slices $p$, required to reach $log_{10}(||F||_2) < -6$ and $log_{10}(||h||_2) < -5$ and with the discretizing of $B(u)$ in $O(\Delta t^2)$. $log_{10}(min/max(\kappa_2(\bar{S}_\Gamma^{(i)})))$ (respectively $log_{10}(min/max(\kappa_2(M)))$ refers to the minimum or maximum of the condition number of the local Schur complement of the time slices 2 to $p-1$ (respectively of the Lagrange multipliers system) over the Newton iterations.

do not lead to convergence, excepted for $p = 2$ where the interface system is reduced to one point. This lack of convergence is due to local Jacobian matrices that become singular because $g(t,y)$ is constant. However, T.C. (10c)(12c) lead to convergence in the same number of Newton iterations as for one time domain except for $p = 100$, where the condition number of local Schur complements increases.

## 5 conclusions

We have extended the time domain decomposition that transforms the IVP into a BVP in order to introduce a Dual Schur complement inside the Newton method. This allows the Newton iterative solution to satisfy the continuity constraints at the time slices boundaries. Nevertheless, in this nonlinear framework the right transmission conditions for defining the local functions on time slices are those involving the flux even if the number of time slices that can be used reaches a limit due to the bad condition number of the local Schur complements.

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## References

1. Alfredo Bellen and Marino Zennaro. Parallel algorithms for initial value problems for difference and differential equations. *J. Comput. Appl. Math.*, 25(3):341–350, 1989.
2. Ernst Hairer, Christian Lubich, and Gerhard Wanner. *Geometric numerical integration*, volume 31 of *Springer Series in Computational Mathematics*. Springer-Verlag, Berlin, second edition, 2006. Structure-preserving algorithms for ordinary differential equations.
3. Herbert B. Keller. Existence theory for two point boundary value problems. *Bull. Amer. Math. Soc.*, 72:728–731, 1966.
4. Patrice Linel and Damien Tromeur-Dervout. Aitken-Schwarz and Schur complement methods for time domain decomposition. In *Parallel Computing: From Multicores and GPU's to Petascale*, volume 19 of *Advances in Parallel Computing*, pages 75–82. IOS Press, 2010.
5. Patrice Linel and Damien Tromeur-Dervout. Une méthode de décomposition en temps avec des schémas d'intégration réversible pour la résolution de systèmes d'équations différentielles ordinaires. *C. R. Math. Acad. Sci. Paris*, 349(15-16):911–914, 2011.
6. Patrice Linel and Damien Tromeur-Dervout. Analysis of the time-Schwarz DDM on the heat PDE. *Computers & Fluids*, 80(0):94 – 101, 2013.
7. Patrice Linel and Damien Tromeur-Dervout. Nonlinear Transmission Conditions for time Domain Decomposition Method. In J. Erhel, M.J. Gander, L. Halpern, G. Pichot, T. Sassi, and O. Widlund, editors, *Domain Decomposition Methods in Science and Engineering XXI*, volume 98 of *Lecture Notes in Computational Science and Engineering*, pages 807–814. Springer, 2014.