An Alternative Coarse Space Method for Overlapping Schwarz Preconditioners for Raviart-Thomas Vector Fields

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Summary. The purpose of this paper is to introduce an overlapping Schwarz method for vector field problems discretized with the lowest order Raviart-Thomas finite elements. The coarse component of the preconditioner is based on energy-minimizing discrete harmonic extensions and the local components consist of traditional solvers on overlapping subdomains. The approach has a couple of benefits compared to the previous methods. The algorithm can be implemented in an algebraic manner. Moreover, the method leads to a condition number independent of the values and jumps of the coefficients across the interface between the substructures. Supporting numerical examples to demonstrate the effectiveness are also presented.

1 Introduction

Domain decomposition methods can be categorized in two classes: overlapping Schwarz methods with overlapping subdomains and iterative substructuring methods with nonoverlapping subdomains. In this paper, we consider two level overlapping Schwarz algorithms. Such methods were originally developed for scalar elliptic problems; see [11, 15] and references therein. Later these methods have also been considered for solving vector fields problems posed in $H(\mathrm{div})$ and $H(\mathbf{curl})$; see [1, 9, 13]. Other types of algorithms, such as multigrid methods, classical iterative substructuring methods, balancing Neumann-Neumann, and FETI methods, have also been suggested in [3, 8, 12, 14, 16, 17]. Many nonoverlapping methods have been studied for discontinuous coefficients cases for vector fields problems. However, only few methods were introduced for the overlapping Schwarz methods in case of coefficients which have jumps.

In the domain decomposition theory, methods can often provide good scalability, i.e., the condition number of the preconditioned system will depend only on the size of the subdomain problems and not on any other parameters, e.g., the number of subdomains and jumps of the coefficients. For the purpose of handling the discontinuity, we borrow the advanced coarse space techniques of [6, 7] based on discrete harmonic extensions of coarse trace spaces developed for almost incompressible elasticity.

The rest part of this paper is organized as follows. We introduce a model problem and its finite element approximation in Sect. 2. In Sects. 3 and 4, we recall the

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overlapping Schwarz method and we suggest the alternative coarse algorithm, respectively. We next present the numerical results in Sect. 5. Finally, the conclusion of this paper is given in Sect. 6.

2 Discretized Problem

We consider the following second order partial differential equation for vector field problem posed in $H(\mathrm{div})$ in a bounded polyhedral domain Ω with a homogeneous boundary condition:

$$\begin{aligned} L\mathbf{u} := -\mathbf{grad}\,(\alpha\,\mathrm{div}\,\mathbf{u}) + \beta\,\mathbf{u} &= \mathbf{f} \text{ in } \Omega, \\ \mathbf{u}\cdot\mathbf{n} &= 0 \text{ on } \partial\Omega. \end{aligned} \tag{1}$$

Here we have positive coefficients $\alpha, \beta \in L^\infty(\Omega)$ and assume that $\mathbf{f}$ is in $(L^2(\Omega))^3$. The main focus of our work is on the coefficients α and β which have jumps across between the substructures.

The model problem (1) has many important applications, such as a mixed and least-squares formulation of certain types of second order partial differential equations [5, 17]. There are other types of applications related to $H(\mathrm{div})$, e.g., iterative solvers for the Reissner-Mindlin plate and the sequential regularization method for the Navier-Stokes equations. For more detail, see [2, 10].

We next consider a variational formulation of (1):

$$\mathbf{a}(\mathbf{u},\mathbf{v}) := \int_\Omega \alpha\,\mathrm{div}\,\mathbf{u}\,\mathrm{div}\,\mathbf{v}\,dx + \beta\,\mathbf{u}\cdot\mathbf{v}\,dx = \int_\Omega \mathbf{f}\cdot\mathbf{v}\,dx,\ \mathbf{v}\in H_0(\mathrm{div};\Omega). \tag{2}$$

We consider the lowest order Raviart-Thomas elements, conforming in $H(\mathrm{div})$, to obtain a discretized problem; see [4, Chap. 3]. We note that the degrees of freedom of the Raviart-Thomas elements are defined by the average values of the normal components over the faces.

Let us consider the variational problem (2). Restricting to the finite element space of the lowest order Raviart-Thomas elements with shape regular and quasi-uniform meshes, we obtain the following linear system:

$$Au = f, \tag{3}$$

where the matrix A is a stiffness matrix, u is a vector of degrees of freedom, and f is a known vector obtained from $\mathbf{f}$. We note that A is symmetric and positive definite.

3 Overlapping Schwarz Preconditioner

We consider a decomposition of the domain Ω into N nonoverlapping subdomains Ω_i, $i = 1,\cdots,N$. We next introduce extended subregions Ω_i' obtained from Ω_i by adding layers of elements and the interface Γ which is given by

$$\Gamma = \left(\bigcup_{i=0}^{N} \partial \Omega_i \right) \backslash \partial \Omega.$$

We consider a two-level overlapping Schwarz algorithm to solve the linear system (3). An overlapping Schwarz preconditioner usually has the following form:

$$P^{-1} = R_0^T A_0^{-1} R_0 + \sum_{i=1}^{N} R_i^T A_i^{-1} R_i, \tag{4}$$

where A_0 is the matrix of the global coarse problem, the A_i's are obtained from local subproblems related to the extended subdomains Ω_i', and R_0 and R_i's are restriction operators to the coarse space and local spaces, respectively; see [11, 15] for more details.

In [9, 13], model problems were designed for constant coefficients and convex domains to analyze the methods. In our work, we use more general assumptions: convex subdomains and coefficients which have jumps across the interface Γ.

In order to deal with this situation, we consider an alternative coarse space approach instead of traditional coarse interpolations. The basis functions for the new algorithm are based on energy-minimizing discrete harmonic extensions with given interface values. We use the corresponding discrete harmonic extensions of the boundary values of standard basis functions to construct new basis functions. We remark that this process can be performed locally and in parallel due to the fact that the basis functions are supported in just two subdomains. We also note that we do not need any coarse triangulation and this work can be done algebraically. With new alternative basis functions, we obtain the operator R_0 which defines the new basis and the matrix $A_0 = R_0 A R_0^T$ associated with the global coarse problem.

For the local components, we follow the traditional way. Each R_i is a rectangular matrix with elements equal to 0 and 1 and provides the indices relevant to an individual extended subdomain Ω_i'. Each $A_i = R_i A R_i^T$ is just the principal minor of the original stiffness matrix A defined by R_i. By using these matrices, we can build the local component $\sum_{i=1}^{N} R_i^T A_i^{-1} R_i$ of the Schwarz preconditioner.

4 The Coarse Component

In this section, we explain our approach in detail. We focus on the restriction operator R_0 onto the coarse space. Before we consider the alternative method, we introduce the conventional method in [9, 13]. The restriction operator is obtained by the interpolation from the subspaces defining the coarse component to the global space. More precisely, R_0 are exactly the coefficients obtained by interpolating the traditional coarse basis functions onto the fine mesh. We note that we need geometric information, e.g., coordinate information, to construct R_0.

Instead of the conventional coarse basis, we will use discrete harmonic extensions to define the new coarse basis functions. We first consider two adjacent subdomains Ω_i and Ω_j. We then have a coarse face $F_{ij} = \partial \Omega_i \cap \partial \Omega_j$. We note that each coarse degree of freedom of our coarse component is related to each coarse face. Let u denote the vector of degrees of freedom for the original problem. Similarly, we consider the vectors of degrees of freedom $u_I^{(i)}$, $u_I^{(j)}$, and $u_{F_{ij}}$ associated with $\Omega_i \backslash \Gamma$, $\Omega_j \backslash \Gamma$, and F_{ij}, respectively. We then have restriction matrices $R_I^{(i)}$, $R_I^{(j)}$, and $R_{F_{ij}}$, i.e., $u_I^{(i)} = R_I^{(i)} u$, $u_I^{(j)} = R_I^{(j)} u$, and $u_{F_{ij}} = R_{F_{ij}} u$. We note that each restriction matrix has only one nonzero entry of unity per each row. We next introduce a submatrix of the stiffness matrix A. It corresponds to the two subdomains which have F_{ij} in common:

$$\begin{bmatrix} A_{II}^{(i)} & 0 & A_{IF_{ij}}^{(i)} \\ 0 & A_{II}^{(j)} & A_{IF_{ij}}^{(j)} \\ A_{F_{ij}I}^{(i)} & A_{F_{ij}I}^{(j)} & A_{F_{ij}F_{ij}} \end{bmatrix}.$$

We choose $u_{F_{ij}}^T = [1, 1, \cdots, 1]$ and introduce the local subproblems $A_{II}^{(i)} u_I^{(i)} + A_{IF_{ij}}^{(i)} u_{F_{ij}} = 0$ and $A_{II}^{(j)} u_I^{(j)} + A_{IF_{ij}}^{(j)} u_{F_{ij}} = 0$ to consider discrete harmonic extensions; see [15, Chap. 4.4]. Then, $u_I^{(i)}$ and $u_I^{(j)}$ are completely determined by $u_{F_{ij}}$, i.e., $u_I^{(i)} = E_i u_{F_{ij}}$ and $u_I^{(j)} = E_j u_{F_{ij}}$, where $E_i := -A_{II}^{(i)^{-1}} A_{IF_{ij}}^{(i)}$ and $E_j := -A_{II}^{(j)^{-1}} A_{IF_{ij}}^{(j)}$. We then obtain a coarse basis $u_{ij} = R_I^{(i)^T} u_I^{(i)} + R_I^{(j)^T} u_I^{(j)} + R_{F_{ij}}^T u_{F_{ij}}$ corresponding to F_{ij}. We can then construct the following form of our coarse interpolation matrix R_0 after the similar process:

$$R_0 := \begin{bmatrix} \vdots \\ - \; u_{ij}^T \; - \\ \vdots \end{bmatrix}.$$

As we mentioned earlier, we can obtain the coarse matrix A_0 by the Galerkin product $R_0 A R_0^T$. We remark that our alternative approach can be implemented in an algebraic manner and in parallel. However, we need to solve additional local Dirichlet-type subproblems to construct the coarse component compared to the conventional methods.

5 Numerical Experiments

We apply the overlapping Schwarz method with the energy-minimizing coarse space to our model problem. We use $\Omega = (0,1) \times (0,1) \times (0,1)$ and the lowest order hexahedral Raviart-Thomas elements. We decompose the domain into $N \times N \times N$ identical subdomains. In each subdomain, we assume that the coefficients α and β are constant. We consider cases where the coefficients have jumps across the interface between the subdomains, in particular, a checkerboard distribution pattern. Each subdomain Ω_i has side length $H = 1/N$ and each mesh cube has h as a minimum side length. We also introduce extended subdomains whose boundaries do not cut any mesh elements with an overlap parameter δ between subdomains. We use the preconditioned conjugate gradient method to solve the preconditioned linear system

$$P^{-1}Au = P^{-1}f. \tag{5}$$

We stop the iteration when the residual l_2-norm has been reduced by a factor of 10^{-6}.

We perform two different kinds of experiments. We first fix the overlap parameter H/δ and vary H/h. We next fix the size of H/h and use various size of H/δ. We report the condition numbers estimated by the conjugate gradient method and the number of iterations. Tables 1 and 3 show the first results and Tables 2 and 4 show the results of the second experiments.

In the first set of experiments, we see that the condition numbers and the iteration counts do not depend on the size of H/h. In the second set, we can conclude that the condition numbers grow linearly with H/δ. For both cases, the condition numbers and iteration counts are quite independent of coefficients and the jumps of coefficients between the subdomains. AQ1

Table 1. Condition numbers and iteration counts. $\alpha_i = 1$ or specified values as indicated in a checkerboard pattern, $\beta_i \equiv 1$, $\frac{H}{\delta} = 4$, $H = \frac{1}{3}$, and $h = \frac{1}{12}, \frac{1}{24}, \frac{1}{48}$

	$\alpha_i = 0.01$		$\alpha_i = 0.1$		$\alpha_i = 1$		$\alpha_i = 10$		$\alpha_i = 100$	
$\frac{H}{h}$	cond	iters	cond	iters	cond	iters	cond	iters	cond	iters
4	8.23	15	8.90	16	9.16	17	8.92	16	8.25	15
8	8.39	16	9.01	17	9.20	18	9.00	17	8.28	16
16	8.23	16	8.99	17	9.22	19	8.98	17	8.28	16

Table 2. Condition numbers and iteration counts. $\alpha_i = 1$ or specified values as indicated in a checkerboard pattern, $\beta_i \equiv 1$, $\frac{H}{h} = 16$, $H = \frac{1}{3}$, and $h = \frac{1}{48}$

	$\alpha_i = 0.01$		$\alpha_i = 0.1$		$\alpha_i = 1$		$\alpha_i = 10$		$\alpha_i = 100$	
$\frac{H}{\delta}$	cond	iters	cond	iters	cond	iters	cond	iters	cond	iters
4	8.23	16	8.99	17	9.22	19	8.98	17	8.28	16
8	10.86	16	13.27	18	14.06	22	14.16	18	14.10	16
16	16.22	18	22.94	22	25.03	24	25.30	22	25.32	20

6 Conclusion

An alternative coarse space technique based on energy-minimizing discrete harmonic extensions for overlapping Schwarz algorithm for vector field problems posed in

Table 3. Condition numbers and iteration counts. $\beta_i = 1$ or specified values as indicated in a checkerboard pattern, $\alpha_i \equiv 1$, $\frac{H}{\delta} = 4$, $H = \frac{1}{3}$, and $h = \frac{1}{12}, \frac{1}{24}, \frac{1}{48}$

	$\beta_i = 0.01$		$\beta_i = 0.1$		$\beta_i = 1$		$\beta_i = 10$		$\beta_i = 100$	
$\frac{H}{h}$	cond	iters	cond	iters	cond	iters	cond	iters	cond	iters
4	8.18	15	8.36	16	9.16	17	8.68	17	8.36	16
8	8.18	17	8.46	18	9.20	18	8.65	18	8.37	18
16	8.18	17	8.45	18	9.22	19	8.62	18	8.37	18

Table 4. Condition numbers and iteration counts. $\beta_i = 1$ or specified values as indicated in a checkerboard pattern, $\alpha_i \equiv 1$, $\frac{H}{h} = 16$, $H = \frac{1}{3}$, and $h = \frac{1}{48}$

	$\beta_i = 0.01$		$\beta_i = 0.1$		$\beta_i = 1$		$\beta_i = 10$		$\beta_i = 100$	
$\frac{H}{\delta}$	cond	iters	cond	iters	cond	iters	cond	iters	cond	iters
4	8.18	17	8.45	18	9.22	19	8.62	18	8.37	18
8	8.50	17	9.98	18	14.06	22	13.48	21	9.43	19
16	9.34	17	13.13	21	25.03	24	24.79	22	12.56	19

$H(\mathrm{div})$ has been introduced and implemented. The numerical results show the usefulness of our method even in the presence of jumps of the coefficients between the substructures.

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