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23. Two iterative substructuring methods for Maxwell's equations with discontinuous coefficients in two dimensions

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Introduction

In this paper, we present some numerical results for a Balancing and a FETI method for the solution of a linear system arising from the edge element approximation of a vector field problem in two dimensions. The two methods are presented as projected preconditioned conjugate algorithms and give comparable performances in our tests. Our numerical results show that their condition number is independent of the number of substructures and grows only polylogarithmically with the number of unknowns associated with individual substructures. It is also independent of the jumps of both coefficients of the original problem.

We consider the following problem: Find $\mathbf{u} \in H_0(\mathrm{curl}\,;\Omega)$, such that

$$a(\mathbf{u},\mathbf{v}) = \int_\Omega \mathbf{f}\cdot\mathbf{v}\,dx, \quad \mathbf{v} \in H_0(\mathrm{curl}\,;\Omega), \tag{1}$$

where the bilinear form $a(\cdot,\cdot)$ is defined as

$$a(\mathbf{u},\mathbf{v}) := \int_\Omega \left(a\,\mathrm{curl}\,\mathbf{u}\,\mathrm{curl}\,\mathbf{v} + b\,\mathbf{u}\cdot\mathbf{v}\right)\,dx,$$

and $\mathbf{f} \in L^2(\Omega)^2$. Here, Ω is a bounded, open, connected polygon in $\mathbb{R}^2$, $H(\mathrm{curl}\,;\Omega)$ is the space of vectors in $L^2(\Omega)^2$, with curl in $L^2(\Omega)$, and $H_0(\mathrm{curl}\,;\Omega)$ its subspace of vectors with vanishing tangential component on $\partial\Omega$. The coefficients a and b are positive functions in $L^\infty(\Omega)$ bounded away from zero.

Finite element functions

For the discretization of problem (1), we consider a conforming triangulation $\mathcal{T}_h$ of Ω, of maximum diameter h, consisting of triangles or rectangles. We then define U as the space of edge elements of lowest degree, defined on $\mathcal{T}_h$, originally introduced in [Nő80]. Let $\mathcal{E}_h$ be the set of edges of $\mathcal{T}_h$. We recall that the tangential components of the vectors in U are constant along the edges of $\mathcal{T}_h$ and that these constant values can be chosen as the degrees of freedom in U.

We then consider a non–overlapping partition of the domain Ω, consisting of subdomains, also called substructures, $\mathcal{F}_H = \{\Omega_i |\, i = 1,\ldots,N\}$. The substructures are

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connected polygonal domains the boundaries of which do not cut through the elements, and H is the maximum of their diameters. Let $\mathbf{t}_i$ be the unit tangent to $\partial\Omega_i$, having counterclockwise direction and restricted to $\partial\Omega_i \setminus \partial\Omega$. We will employ these unit vectors to define the coarse spaces of our algorithms. We also define the interface Γ as

$$\Gamma := \bigcup_{i=1}^{N} \partial\Omega_i \setminus \partial\Omega.$$

We remark that we only present numerical results for uniform meshes in this paper, but that our algorithms can be defined for more general cases. In particular, a theoretical bound for a FETI method, which is valid for triangulations that are shape–regular and locally quasi uniform, was proven in [TK99]. In the following, we assume, for simplicity, that the coefficient b is constant on each substructure and equal to b_i.

Given a substructure Ω_i, we define U_i as the space of restrictions of vectors in U to the Ω_i. We also define the local spaces W_i of tangential vectors as

$$W_i := \{(\mathbf{u}_i \cdot \mathbf{t}_i)\,\mathbf{t}_i \text{ restricted to } \partial\Omega_i \setminus \partial\Omega \;\; | \; \mathbf{u}_i \in U_i\}.$$

The vectors in W_i are uniquely determined by the degrees of freedom on $\partial\Omega_i$. In the following, the column vector of degrees of freedom of $\mathbf{u}_i \in W_i$ will be denoted by u_i, and it will be convenient to use the same notation for spaces of vectors and the corresponding spaces of degrees of freedom.

The finite element discretization of (1) gives rise to a symmetric, positive definite linear system. The degrees of freedom inside the substructures and on $\partial\Omega$ only belong to one substructure and can be eliminated in parallel by block Gaussian elimination. We are then left with a linear system involving only the degrees of freedom on Γ. Let S_i be the local Schur complement relative to the degrees of freedom on $\partial\Omega_i \setminus \partial\Omega$

$$S_i : W_i \longrightarrow W_i.$$

If a local vector on Ω_i is divided into two subvectors, of degrees of freedom corresponding to edges inside Ω_i and on $\partial\Omega_i \setminus \partial\Omega$, respectively, the local stiffness matrix of A_i can be written as

$$A_i = \begin{bmatrix} A_i^{II} & A_i^{IB} \\ A_i^{BI} & A_i^{BB} \end{bmatrix},$$

and the Schur complement S_i is defined as

$$S_i := A_i^{BB} - A_i^{BI} \left(A_i^{II}\right)^{-1} A_i^{IB}.$$

Before introducing our algorithms, we need to define a set of local scaling functions. These functions are constructed with the values of the coefficient b and ensure that the condition number of our iterative methods is independent of the jumps of *both* coefficients. For a substructure Ω_i, we define a piecewise constant function $\mu_i^\dagger$ on $\partial\Omega_i \setminus \partial\Omega$ such that

$$\mu_{i\,|_e}^\dagger \equiv \frac{b_i^\delta}{b_i^\delta + b_k^\delta}, \quad e \subset \partial\Omega_i \cap \partial\Omega_k, \quad e \in \mathcal{E}_h,$$

where $\delta \geq 1/2$ is arbitrary but fixed. Let D_i be the diagonal matrix that represents the multiplication of vectors in W_i by $\mu_i^\dagger$.

Conjugate Gradient algorithms

The two methods that we consider can be described as projected preconditioned conjugate gradient (PPCG) algorithms. We suppose that we are looking for the solution of a symmetric, positive definite linear system

$$Fz = d, \quad z \in V, \tag{2}$$

arising from a finite element discretization of an elliptic problem.

We first introduce a suitable subspace $V_0 \subset V$, of low dimension K, that will play the role of a *coarse space*, and define P_0 as the projection onto V_0 that is orthogonal with respect to the scalar product induced by F. The operator

$$P := I - P_0,$$

is also an orthogonal projection and, if

$$V = V_0 \oplus V^\perp,$$

we have that $Range(P) = V^\perp$. Let $z_0 = P_0 z$ be the projection of the solution z onto V_0.

We consider the following preconditioned system

$$PMP^t\, Fz = PMP^t\, d, \quad z \in z_0 + V^\perp, \tag{3}$$

where the preconditioner M has the form

$$M := \sum_{i=1}^{N} M_i,$$

and the application of the *local component* M_i involves the solution of a local problem on the substructure Ω_i. Here, P^t denotes the transpose of the matrix P. Recalling the definition of P_0, we see that $P^t \neq P$, in general.

A full description of the conjugate gradient method applied to Equation (3) can be found in [FCM95, Tos00, TK99]. Here, we only remark that the action of the projection P on a vector can be evaluated at the expense of applying the matrix F and of solving a coarse problem of dimension K. Moreover, the action of P^t does not need to be calculated in practice.

A suitable choice of the projection P ensures that the condition number of the corresponding iterative method is independent of the number of substructures and depends only on the ratio $\rho = H/h$, which is a measure of the number of degrees of freedom in each substructure. In addition, a suitable choice of the preconditioner M ensures that the condition number is slowly increasing with ρ and is independent of possibly large jumps of the coefficients.

A Balancing method

The method that we present is a variant of the Neumann–Neumann algorithm introduced and analyzed in [Tos00]: it employs the same preconditioner M, but a different coarse space V_0. In [Tos00], the partition $\mathcal{F}_H$ is required to be a conforming coarse triangulation of Ω and V_0 is the standard edge element space defined on it, while, here, the partition is arbitrary and the basis functions of V_0 are associated to the substructures.

We consider the linear system obtained from the approximation of Problem (1) on the conforming finite element space U and define W as the space of tangential components of the vectors in U on Γ. We note that the restrictions of the vectors in W to $\partial\Omega_i \setminus \partial\Omega$ belong to W_i, for $i = 1, \ldots, N$. After eliminating the variables interior to the substructures, we are left with the system

$$Su = g, \quad u \in W, \tag{4}$$

where S is the global Schur complement matrix relative to Γ and g is the resulting right hand side. We define the operators

$$R_i^t : W_i \longrightarrow W, \quad i = 1, \ldots, N,$$

as the extensions by zero of local vectors in W_i on the whole Γ, and note that the R_i are the restriction operators from W to W_i. We can then write

$$S = \sum_{i=1}^{N} R_i^t S_i R_i.$$

Problem (4) then corresponds to the choice $F = S$, $d = g$, $V = W$, in (2). We define the coarse space as the span of the extensions to Γ of the vectors $\{\mathbf{t}_i\}$:

$$V_0 := span\{R_i^t \, t_i \mid i = 1, \ldots, N\}.$$

It can easily be checked that the dimension of V_0 is equal to the number of substructures minus one.

Following [Tos00], we define the local components of the preconditioner as

$$M_i := R_i^t \, D_i \, S_i^{-1} D_i \, R_i, \quad i = 1, \ldots, N.$$

A FETI method

The method presented in this section was originally developed and analyzed in [TK99]. We define the non–conforming space $\widehat{W}$ as

$$\widehat{W} := \prod_{i=1}^{N} W_i.$$

We note that the vectors in $\widehat{W}$ are in general discontinuous across Γ and, given two substructures, Ω_i and Ω_k, that share a common edge, there are two different fields on $\partial\Omega_i \cap \partial\Omega_k$ that correspond to a vector $\mathbf{u} \in \widehat{W}$. We define the block diagonal matrix

$$\widehat{S} := \mathrm{diag}\{S_1, \, S_2, \, \cdots, S_N\} \, : \, \widehat{W} \longrightarrow \widehat{W}.$$

We can then formulate our finite element problem as a constrained minimization problem: Find $u \in \widehat{W}$, such that

$$\left.\begin{array}{c} \frac{1}{2} u^t \, \widehat{S} \, u - u^t \, g \;\longrightarrow \min \\ B u = 0 \end{array}\right\} \tag{5}$$

where the matrix B evaluates the difference of the corresponding degrees of freedom on Γ and can be written as

$$B = \left[B^{(1)} \; B^{(2)} \; \cdots \; B^{(N)} \right].$$

Here, g is constructed with the local load vectors on the substructures. We then introduce a vector of Lagrange multipliers λ, to enforce the constraints, and obtain a saddle point formulation of (5). After eliminating the primal variable u, we obtain the following equation for the dual variable λ, see [FCM95, TK99],

$$B \widehat{S}^{-1} B^t \lambda = B \widehat{S}^{-1} g, \quad \lambda \in \mathit{Range}(B). \tag{6}$$

We consider a PPCG method for the solution of (6). This corresponds to the choice $F = B\widehat{S}^{-1}B^t$, $d = B\widehat{S}^{-1}g$, $V = \mathit{Range}(B)$, in (2). We note that V is the space of jumps of the tangential vectors in $\widehat{W}$. We then define the coarse space V_0 as a space of scaled jumps of the local vectors $\{\mathbf{t}_i\}$

$$V_0 := \mathit{span}\{B_i \, (I - D_i) \, t_i \mid i = 1, \ldots, N\}.$$

We refer to [TK99, Sect. 5] for additional details and for a discussion of the dimension of V_0. In particular, we note that the vectors $\{t_i\}$ also need to be scaled using the lengths of the edges in $\mathcal{E}_h$ if the mesh $\mathcal{T}_h$ is not uniform.

Following [TK99, KW99], we define the local components of the preconditioner as

$$M_i := (B\widehat{D}^{-1}B^t)^{-1} \, B_i D_i^{-1} \widehat{S}_i D_i^{-1} B_i^t \, (B\widehat{D}^{-1}B^t)^{-1},$$

where $\widehat{D} := \mathrm{diag}\{D_1, \, D_2, \, \cdots, D_N\}$.

Numerical results

We first remark that, for the Balancing method, at each conjugate gradient step, we need to solve one Neumann problem on each substructure for the application of the preconditioner, and two Dirichlet problems for the application of S and P (we recall that P is a projection that is orthogonal with respect to the scalar product induced by $F = S$). Similarly, for the FETI method, at each step, we need to solve two Neumann problems and one Dirichlet problem on each substructure. We refer to [FCM95, Tos00, TK99] for additional comments.

In our numerical tests, we consider the domain $\Omega = (0, 1)^2$ and two uniform triangulations $\mathcal{T}_h$ and $\mathcal{F}_H$. The fine triangulation is made of triangles for the FETI method, and squares for the Balancing method. It consists of $2 * n^2$ triangles and n^2 squares, respectively, with $h = 1/n$. We note that, as opposed to the case of nodal

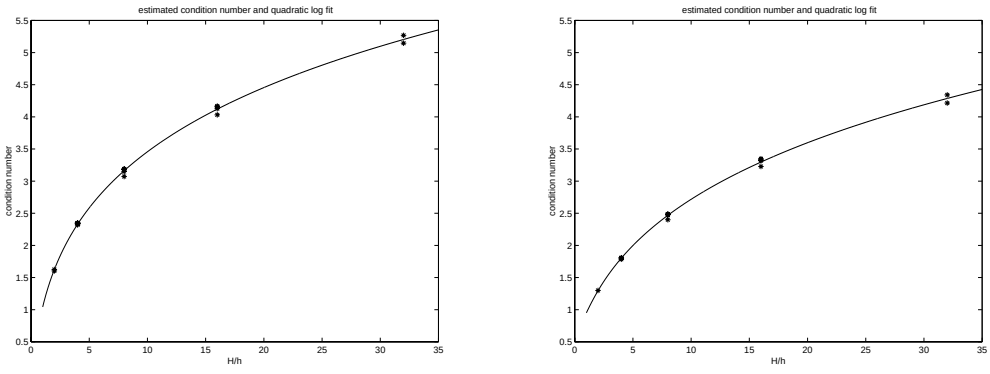

Figure 1: Case with $a = 1$, $b = 1$, $n = 32, 64, 128, 192, 256$. Estimated condition number (asterisk) and least–square second order logarithmic polynomial (solid line), versus $\rho = H/h$ for the Balancing (on the left) and the FETI (on the right) methods.

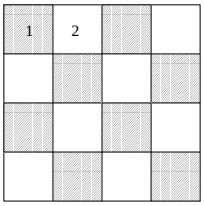

Figure 2: Checkerboard distribution of the coefficients in the unit square.

elements, for a fixed value of n, triangular and rectangular meshes do not give rise to edge element spaces of the same dimension. Nevertheless, the mesh size h and the order of accuracy is the same, see [Ń80], and our comparisons of the two methods are still reasonably fair. The coarse triangulation consists of nc^2 squares which are unions of fine elements, with $H = 1/nc$. The substructures Ω_i are the elements of the coarse triangulation $\mathcal{F}_H$. Throughout, we use the value $\delta = 1/2$.

We first consider a case with constant coefficients and meshes with $n = 32, 64, 128, 192, 256$. Figure 1 shows the estimated condition number (asterisks), for $a = b = 1$, as a function of $\rho = H/h$, for different values of n. The results for the FETI method are taken from [TK99]. For a fixed value of ρ, the condition number is quite insensitive to the dimension of the fine mesh. We have also plotted the best second order logarithmic polynomial least–square fits; our numerical results for both methods are consistent with the bound for the condition number

$$\kappa(PMP^tF) \leq C \left(1 + \log \frac{H}{h}\right)^2,$$

and suggest that this bound is sharp. We note that this bound was proved in [TK99] for the FETI method.

We then consider some cases where the coefficients have jumps. In Table 1, we show some results when the coefficient b has jumps across the substructures. We

b_2, ρ	4	8	16
10^{-4}	15.6 (22)	13.4 (22)	12.1 (22)
10^{-3}	15.1 (21)	13.2 (21)	12.1 (23)
10^{-2}	13.8 (20)	12.5 (21)	11.9 (23)
10^{-1}	10.8 (19)	10.8 (21)	11.5 (22)
1	6.31 (17)	7.55 (19)	10.2 (21)
10	3.87 (13)	5.41 (15)	7.36 (18)
10^2	2.33 (8)	3.12 (10)	3.87 (11)
10^3	3.70 (12)	4.77 (14)	5.56 (16)
10^4	3.96 (14)	4.33 (14)	4.64 (15)
10^5	3.27 (12)	3.55 (13)	4.34 (14)
10^6	2.99 (12)	3.44 (13)	4.28 (14)

4	8	16
4.12 (17)	5.99 (22)	8.42 (26)
4.09 (16)	5.96 (20)	8.37 (25)
4.04 (15)	5.88 (19)	8.25 (23)
3.88 (13)	5.65 (17)	7.91 (21)
3.44 (12)	5.02 (15)	6.99 (18)
2.56 (9)	3.73 (12)	5.16 (14)
1.76 (7)	2.41 (8)	3.10 (10)
2.51 (9)	3.37 (11)	3.99 (12)
2.74 (10)	3.09 (11)	3.51 (11)
2.20 (9)	2.73 (10)	3.35 (11)
2.09 (9)	2.65 (10)	3.34 (12)

Table 1: Checkerboard distribution for b: (b_1, b_2). Estimated condition number and number of CG iterations to obtain a relative preconditioned residual less than 10^{-6} (in parentheses), versus $\rho = H/h$ (columns) and b_2 (rows), for the Balancing (on the left) and the FETI (on the right) methods. Case of $n = 128$, $a = 1$, and $b_1 = 100$.

consider the checkerboard distribution shown in Figure 2, where b is equal to b_1 in the shaded area and to b_2 elsewhere. For a fixed value of $n = 128$, $b_1 = 100$, and $a = 1$, the estimated condition number and the number of iterations in order to obtain a reduction of the norm of the preconditioned residual by a factor 10^{-6}, are shown as a function of $\rho = H/h$ and b_2. For $b_2 = 100$, the coefficient b has a uniform distribution, and this corresponds to a minimum for the condition number and the number of iterations for both methods. When b_2 decreases or increases, the condition number and the number of iterations also increase, but they can still be bounded independently of b_2. We observe that the two methods give comparable iteration counts.

In Table 2, we show some results when the coefficient a has jumps. We consider the checkerboard distribution shown in Figure 2, where a is equal to a_1 in the shaded area and to a_2 elsewhere. For a fixed value of $n = 128$, $a_1 = 0.01$, and $b = 1$, the estimated condition number and the number of iterations are shown as a function of $\rho = H/h$ and a_2. We remark that for $a_2 = 0.01$, the coefficient a has a uniform distribution. For both methods, a slight increase in the number of iterations and the condition number is observed, when a_2 is decreased or increased and when H/h is large.

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a_2, ρ	4	8	16
10^{-7}	2.56 (10)	4.33 (13)	8.02 (17)
10^{-6}	2.56 (10)	4.32 (13)	8.01 (17)
10^{-5}	2.56 (10)	4.30 (13)	7.96 (17)
10^{-4}	2.52 (9)	4.13 (13)	7.49 (16)
10^{-3}	2.39 (9)	3.51 (12)	5.59 (14)
10^{-2}	2.34 (9)	3.16 (11)	4.14 (13)
10^{-1}	2.32 (8)	3.12 (10)	3.87 (12)
1	2.33 (8)	3.12 (10)	3.87 (11)
10	2.34 (8)	3.16 (10)	4.11 (11)
10^2	2.34 (8)	3.16 (10)	4.14 (12)
10^3	2.34 (8)	3.17 (10)	4.14 (12)

4	8	16
2.80 (8)	4.49 (12)	7.29 (15)
2.41 (8)	3.81 (11)	6.21 (14)
1.82 (7)	2.65 (9)	4.05 (11)
1.79 (7)	2.45 (8)	3.23 (10)
1.78 (7)	2.42 (8)	3.07 (9)
1.76 (7)	2.40 (8)	3.25 (10)
1.77 (7)	2.41 (8)	3.10 (10)
1.77 (7)	2.46 (8)	3.26 (10)
1.77 (7)	2.46 (8)	3.26 (10)
1.77 (7)	2.46 (8)	3.26 (10)
1.77 (7)	2.46 (8)	3.26 (10)

Table 2: Checkerboard distribution for a: (a_1, a_2). Estimated condition number and number of CG iterations to obtain a relative preconditioned residual less than 10^{-6} (in parentheses), versus $\rho = H/h$ (columns) and a_2 (rows), for the Balancing (on the left) and the FETI (on the right) methods. Case of $n = 128$, $b = 1$, and $a_1 = 0.01$.

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