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13. The direct method of lines for incompressible material problems on polygon domains

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Introduction

In this paper, we discuss the numerical solutions of the incompressible material problems on a polygon using a semi-discrete method [HH99]. After a suitable transformation of the coordinates, the original boundary value problem (BVP) is reduced to a discontinuous coefficients problem on a rectangle, which is semi-discreted to a BVP of a system of ordinary differential equations (O.D.E's). After solving the BVP of the system by a direct method, the semi-discrete approximation of the original problem is obtained. It's worth to point out that the semi-discrete approximation in form of separable variables naturally possesses the singularity of the original problem. Finally, the numerical examples show that our method is feasible and very effective for solving the incompressible material problems with singularities numerically.

The use of nearly incompressible materials is common in many engineering applications, such as tires, building and bridge bearings, engine mounts, gaskets etc. The natural rubber is the nearly incompressible material, typically the bulk modulus of rubber is several thousand times of the shear modulus. As the material is undergoing plastic deformations, it is nearly incompressible too. We can use the Stokes equations as a model to deal with the incompressible materials. It is also a model for the incompressible fluids. The stress analysis of incompressible materials becomes very significant.

The difficulties for solving the incompressible material problems numerically are: the stress singularity existing at the joint of the interface, the crack-tip or the corner; the incompressibility and the large deformations. To overcome the above difficulties, a great deal of research effort by engineers and mathematicians has been devoted to the development of the FEM(finite element method) for the numerical approximation of incompressible problems. Herrmann [Her65] presented a mixed variational formulation for incompressible isotropic materials. Babuska and Brezzi [Bab73, Bre74] derived the inf-sup condition for the mixed FEM for incompressible problems. Oden et al [OK82, JTOS82] presented general criteria for stability and convergence of mixed and penalty methods(with reduced integration) and applied these criteria to the analysis of elasticity and Stokesian flow problems. Recently, many researchers developed other methods for incompressible problems [AWS95]. For more references, we refer to the paper by Gadala [Gad86]. The singularities of incompressible materials have also been paid attention by researchers [NAHM96]. We know that the singularities at singular points in the incompressible composite material problems are very complex. On each singular point, the singularity is different. Therefore, the standard finite element method and finite difference method can not give satisfied results for incom-

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pressible material problems. Special consideration is usually needed for the numerical approaches to improve the results.

In this paper, we deal with the more general incompressible material problems on a polygon. Suppose that $\overline{\Omega} = \bigcup_{i=1}^{J} \overline{\Omega}_i \subset R^2$ is a J-material wedge with a boundary $\Gamma = \overline{Oa_1} \bigcup \overline{Oa_{J+1}} \bigcup \Gamma_D$ (see Fig.(1)), where O is the origin of the coordinate system, $\overline{Oa_1}$ is parallel to the x_1-axis, the ith material occupies Ω_i, $\Gamma_D = \bigcup_{i=1}^{J} \Gamma_i$ with $\Gamma_i = \overline{a_i a_{i+1}}$, and $\{a_i = (x_1^i, x_2^i), i = 1, 2, \cdots, J+1\}$ denote the vertexes of the polygon Ω, $x_1^i = R_i \cos\theta_i$, $x_2^i = R_i \sin\theta_i$ satisfying

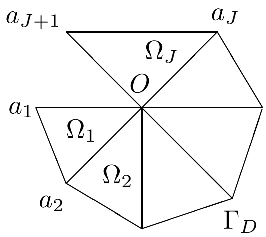

Fig. (1)

$$-\pi = \theta_1 < \theta_2 < \cdots < \theta_{J+1} \le \pi.$$

We consider the following problem of Stokes equations on the J-material wedge Ω:

$$-\mu^i \triangle u^i + \operatorname{grad} p^i = 0, \quad \text{in } \Omega_i, \quad 1 \le i \le J, \tag{1}$$

$$\operatorname{div} u^i = 0, \quad \text{in } \Omega_i, \quad 1 \le i \le J, \tag{2}$$

$$u^i\big|_{\Gamma_i} = f^i, \quad 1 \le i \le J, \tag{3}$$

$$u^{i-1}\big|_{\theta=\theta_i^-} = u^i\big|_{\theta=\theta_i^+}, \qquad \sigma_n^{i-1}\big|_{\theta=\theta_i^-} = \sigma_n^i\big|_{\theta=\theta_i^+}, \quad 2 \le i \le J, \tag{4}$$

$$\sigma_n^1\big|_{\theta=\theta_1} = 0, \qquad \sigma_n^J\big|_{\theta=\theta_{J+1}} = 0, \tag{5}$$

where (r, θ) denotes the polar coordinate in the plane; $u^i = (u_1^i, u_2^i)^T$ denotes the displacement in Ω_i; $\mu^i > 0$ is the Lame constant; $f^i = (f_1^i, f_2^i)^T$ is a given vector valued function on the polygonal line Γ_i; $\sigma_n^{i-1}\big|_{\theta=\theta_i^-} = (\sin\theta_i\ \sigma_{11}^{i-1} - \cos\theta_i\ \sigma_{12}^{i-1}, \sin\theta_i\ \sigma_{21}^{i-1} - \cos\theta_i\ \sigma_{22}^{i-1})^T$, $\sigma_n^i\big|_{\theta=\theta_i^+} = (\sin\theta_i\ \sigma_{11}^i - \cos\theta_i\ \sigma_{12}^i, \sin\theta_i\ \sigma_{21}^i - \cos\theta_i\ \sigma_{22}^i)^T$ denote the normal stress on $\overline{Oa_i}$, and $\sigma^i = (\sigma_{kl}^i)_{2\times 2}$ denote the stress tensor in Ω_i with entries

$$\sigma_{kl}^i = -\delta_{kl} p^i + \mu^i \left(\frac{\partial u_k^i}{\partial x_l} + \frac{\partial u_l^i}{\partial x_k} \right), \quad 1 \le k, l \le 2, \quad 1 \le i \le J.$$

The equivalent variational-differential formulation of problem (1)-(5)

We introduce the transformation of variables on each triangle Ω_i:

$$\left.\begin{aligned} x_1 &= \frac{\rho_i \rho \cos\phi}{\sin(\phi - \alpha_i)} \\ x_2 &= \frac{\rho_i \rho \sin\phi}{\sin(\phi - \alpha_i)} \end{aligned}\right\} \quad \text{for } \theta_i \le \phi \le \theta_{i+1}, \quad 0 \le \rho \le 1; \tag{6}$$

with

$$\left.\begin{aligned}&\sin\alpha_i=\frac{x_2^{i+1}-x_2^i}{|\overline{a_ia_{i+1}}|},\qquad \cos\alpha_i=\frac{x_1^{i+1}-x_1^i}{|\overline{a_ia_{i+1}}|},\\&|\overline{a_ia_{i+1}}|=\sqrt{(x_1^{i+1}-x_1^i)^2+(x_2^{i+1}-x_2^i)^2},\\&\rho_i=x_2^i\cos\alpha_i-x_1^i\sin\alpha_i.\end{aligned}\right\}\quad \text{for}\quad 1\le i\le J. \tag{7}$$

We can show that $\rho_i<0$ and $\sin(\phi-\alpha_i)\neq 0$ for $\theta_i\le\phi\le\theta_{i+1}$. The transformation (6) maps Ω_i onto the rectangle $\widetilde{\Omega}_i=\{(\rho,\phi)|\,\theta_i<\phi<\theta_{i+1},\ 0<\rho<1\}$ and maps segment $\overline{a_ia_{i+1}}$ onto the segment $\{\rho=0,\ \theta_i\le\phi\le\theta_{i+1}\}$ as shown in Fig.(2). Hence the domain Ω is mapped onto $\widetilde{\Omega}=\{(\rho,\phi)|\ -\pi<\phi<\theta_{J+1},\ 0<\rho<1\}$. In the new coordinate (ρ,ϕ), the BVP (1)-(5) is reduced to a discontinuous coefficients problem on the rectangle $\widetilde{\Omega}$. Furthermore, we introduce the following spaces:

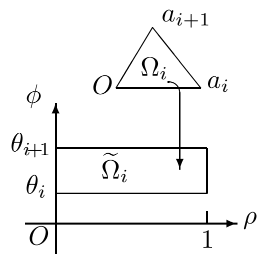

Fig. (2)

$$\begin{aligned}&V_1=\left\{v_1(\phi)\big|\ v_1\in H^1(\theta_1,\theta_{J+1}),\ \text{namely}\ \ v_1,v_1'\in L^2(\theta_1,\theta_{J+1})\right\},\\&U_1=\left\{u_1(\rho,\phi)\Big|\ \text{for fixed}\ \ 0<\rho\le 1,u_1(\rho,\cdot),\frac{\partial u_1}{\partial\rho}(\rho,\cdot),\frac{\partial^2u_1}{\partial\rho^2}(\rho,\cdot)\in V_1\right\},\\&V=V_1\times V_1,\qquad U=U_1\times U_1,\\&Q=\left\{q(\phi)\big|\ q\in L^2(\theta_1,\theta_{J+1})\right\},\\&S=\left\{p(\rho,\phi)\big|\ \text{for fixed}\ \ 0<\rho\le 1,p(\rho,\cdot)\in Q\right\}.\end{aligned}$$

Then the BVP (1)–(5) is equivalent to the following variational-differential problem:

$$\left.\begin{aligned}&\text{Find}\ \ (u,p)\in U\times S,\ \text{such that}\\&-\left(\frac{d}{d\rho}\rho\frac{d}{d\rho}\right)a_2(u,v)+\frac{d}{d\rho}a_1(u,v)+\frac{1}{\rho}a_0(u,v)\\&\qquad\qquad-\left(\frac{d}{d\rho}\rho\right)b_1(p,v)+b_0(p,v)=0,\quad \forall v\in V,\ 0<\rho<1;\\&\rho\frac{d}{d\rho}b_1(q,u)+b_0(q,u)=0,\qquad\qquad\qquad \forall q\in Q,\ 0<\rho<1;\\&u\Big|_{\rho=1}=\widetilde{f},\qquad u\ \text{is bounded , when}\ \rho\to 0;\end{aligned}\right\} \tag{8}$$

where

$$
\begin{aligned}
a_2(u,v) &= \sum_{i=1}^{n} \int_{\theta_i}^{\theta_{i+1}} \frac{\mu^i}{\sin^2(\phi-\alpha_i)} (u^i)^T\, \mathcal{K}_1^i(\alpha_i)\, v^i\, d\phi, \\
a_1(u,v) &= \sum_{i=1}^{n} \int_{\theta_i}^{\theta_{i+1}} \frac{\mu^i}{\sin(\phi-\alpha_i)} \left[\left(\frac{\partial u^i}{\partial \phi} \right)^T \mathcal{K}_2^i\, v^i - (u^i)^T (\mathcal{K}_2^i)^T \frac{dv^i}{d\phi} \right] d\phi, \\
a_0(u,v) &= -\sum_{i=1}^{n} \int_{\theta_i}^{\theta_{i+1}} \mu^i \left(\frac{\partial u^i}{\partial \phi} \right)^T \mathcal{K}_1^i(\phi)\, \frac{dv^i}{d\phi}\, d\phi; \\
b_1(q,v) &= \sum_{i=1}^{n} \int_{\theta_i}^{\theta_{i+1}} \frac{\rho_i q^i}{\sin^2(\phi-\alpha_i)} (\sin\alpha_i v_1^i - \cos\alpha_i v_2^i)\, d\phi, \\
b_0(q,v) &= \sum_{i=1}^{n} \int_{\theta_i}^{\theta_{i+1}} \frac{\rho_i q^i}{\sin(\phi-\alpha_i)} \left(\sin\alpha_i \frac{\partial v_1^i}{\partial \phi} - \cos\alpha_i \frac{\partial v_2^i}{\partial \phi} \right) d\phi,
\end{aligned}
$$

with

$$
\mathcal{K}_1^i(\psi) = \begin{pmatrix} 1+\sin^2\psi & -\dfrac{\sin 2\psi}{2} \\ -\dfrac{\sin 2\psi}{2} & 1+\cos^2\psi \end{pmatrix},
$$

$$
\mathcal{K}_2^i = \begin{pmatrix} \cos(\phi-\alpha_i)+\sin\phi\sin\alpha_i & -\sin\phi\cos\alpha_i \\ -\cos\phi\sin\alpha_i & \cos(\phi-\alpha_i)+\cos\phi\cos\alpha_i \end{pmatrix};
$$

and v^i denotes the restriction of v on $[\,\theta_i,\,\theta_{i+1}]$.

The numerical solution of the variational-differential problem (8)

Suppose that

$$
-\pi = \phi_1 < \phi_2 < \cdots < \phi_{M+1} = \theta_{J+1} \tag{9}
$$

is a partition of the interval $I \equiv [-\pi, \theta_{J+1}]$, such that each of $\{\theta_i\}_{i=1}^n$ is a node of this partition, namely for each θ_i there is a $\phi_j = \theta_i$. Let $h = \max\limits_{1\le j\le M} (\phi_{j+1}-\phi_j)$,

$$
Q_h = \left\{ q^h(\phi) \,\middle|\, q^h \in C^0(I),\ q^h\big|_{[\phi_j,\phi_{j+1}]} \in P_1([\phi_j,\phi_{j+1}]),\ 1 \le j \le M \right\},
$$

$$
S_h = \left\{ p^h(\rho,\phi) \mid \text{for the fixed } 0<\rho\le 1, \quad p^h(\rho,.) \in Q_h \right\}.
$$

Assume that $\{\Phi_j(\phi),\ j=1,2,\cdots,M+1\}$ is a basis of Q_h such that $\Phi_j(\phi_i) = \delta_{ij}$, $1 \le i,j \le M+1$. Furthermore, we refine the partition (9)

$$
-\pi = \phi_1 < \phi_{3/2} < \phi_2 < \cdots < \phi_{M+1/2} < \phi_{M+1} = \theta_{J+1}. \tag{10}
$$

We use quadratic elements to construct the space

$$
V_1^h = \left\{ v_1^h(\phi) \,\middle|\, v_1^h \in C^0(I),\ v_1^h\big|_{[\phi_j,\phi_{j+1}]} \in P_2([\phi_j,\phi_{j+1}]),\ 1 \le j \le M \right\}.
$$

Let $\{\psi_1(\phi),\ \psi_{3/2}(\phi),\ \psi_2(\phi),\ \cdots,\ \psi_M(\phi),\ \psi_{M+1/2}(\phi),\ \psi_{M+1}(\phi)\}$ is a basis of the finite dimensional space V_1^h such that

$$\begin{array}{lll} \psi_j(\phi_i)=\delta_{ij}, & 1\le i\le M+1, & 1\le j\le M+1; \\ \psi_j(\phi_{i+1/2})=0, & 1\le i\le M, & 1\le j\le M+1; \\ \psi_{j+1/2}(\phi_i)=0, & 1\le i\le M+1, & 1\le j\le M; \\ \psi_{j+1/2}(\phi_{i+1/2})=\delta_{ij}, & 1\le i\le M, & 1\le j\le M. \end{array}$$

In addition, we introduce:

$$\begin{array}{rcl} U_1^h &=& \left\{u_1^h(\rho,\phi)\mid \text{for the fixed } 0<\rho\le 1,\quad u_1^h(\rho,.)\in V_1^h\right\}, \\ V_h &=& V_1^h\times V_1^h, \qquad U_h \;=\; U_1^h\times U_1^h. \\ \Phi(\phi) &=& \begin{pmatrix} \Phi_1(\phi) & \Phi_2(\phi) & \cdots & \Phi_{M+1}(\phi) \end{pmatrix}^T, \end{array}$$

$$\Psi(\phi)=\begin{pmatrix} \psi_1(\phi) & 0 & \psi_{3/2}(\phi) & 0 & \cdots & \cdots & \psi_{M+1}(\phi) & 0 \\ 0 & \psi_1(\phi) & 0 & \psi_{3/2}(\phi) & \cdots & \cdots & 0 & \psi_{M+1}(\phi) \end{pmatrix}^T.$$

For $p^h(\rho,\phi)\in S_h$, $u^h(\rho,\phi)\in U_h$, and $\widetilde{f}^h(\phi)\in V_h$ is the interpolating function of $\widetilde{f}$ in space V_h, we have

$$\left.\begin{array}{rcl} p^h(\rho,\phi) &=& \Phi^T(\phi)\stackrel{\wedge}{P}_h(\rho), \\ u^h(\rho,\phi) &=& \Psi^T(\phi)\stackrel{\wedge}{U}_h(\rho), \\ \widetilde{f}^h(\phi) &=& \Psi^T(\phi)F\ , \end{array}\right\} \tag{11}$$

where

$$\stackrel{\wedge}{P}_h(\rho)=\left(p^h(\rho,\phi_1),p^h(\rho,\phi_2),\cdots,p^h(\rho,\phi_{M+1})\right)^T\ ,$$

$$\stackrel{\wedge}{U}_h(\rho)=\left(u_1^h(\rho,\phi_1),u_2^h(\rho,\phi_1),u_1^h(\rho,\phi_{\frac{3}{2}}),u_2^h(\rho,\phi_{\frac{3}{2}}),\cdots,u_1^h(\rho,\phi_{M+1}),u_2^h(\rho,\phi_{M+1})\right)^T,$$

$$F=\left(\widetilde{f}_1(\phi_1),\widetilde{f}_2(\phi_1),\widetilde{f}_1(\phi_{\frac{3}{2}}),\widetilde{f}_2(\phi_{\frac{3}{2}}),\cdots,\widetilde{f}_1(\phi_{M+1}),\widetilde{f}_2(\phi_{M+1})\right)^T.$$

Then we have the numerical approximation of the problem (8):

$$\left.\begin{array}{l} \text{Find } (u^h,p^h)\in U_h\times S_h, \text{ such that} \\ -\left(\dfrac{d}{d\rho}\rho\dfrac{d}{d\rho}\right)a_2(u^h,v^h)+\dfrac{d}{d\rho}a_1(u^h,v^h)+\dfrac{1}{\rho}a_0(u^h,v^h) \\ \qquad -\left(\dfrac{d}{d\rho}\rho\right)b_1(p^h,v^h)+b_0(p^h,v^h)=0, \quad \forall v^h\in V_h,\ 0<\rho<1; \\ \rho\dfrac{d}{d\rho}b_1(q^h,u^h)+b_0(q^h,u^h)=0, \qquad\qquad \forall q^h\in Q_h,\ 0<\rho<1; \\ u^h\Big|_{\rho=1}=\widetilde{f}^h, \qquad u^h \text{ is bounded , when } \rho\to 0. \end{array}\right\} \tag{12}$$

Using (11), the discrete variational-differential problem (12) is equivalent to a BVP of a system of O.D.E's. We can reduce the BVP of the system of O.D.E's to an eigenvalue problem. After solving the eigenvalue problem numerically, we obtain neither more nor less than $4M+2$ eigenvalues λ_j^h $(j=1,2,\cdots,4M+2)$ with non-negative real part corresponding to the eigenvectors $(\zeta_j,\xi_j,\eta_j)^T$, $j=1,2,\cdots,4M+2$, where $\lambda_1^h=\lambda_2^h=0$, $\xi_1=(1,0,\cdots,1,0)^T\in R^{4M+2}$, $\zeta_1=0$, $\xi_2=(0,1,\cdots,0,1)^T\in R^{4M+2}$, $\zeta_2=0$. Particularly we assume λ_j^h $(1\le j\le 2m)$ are real eigenvalues and λ_j^h $(2m+1\le j\le 4M+2)$ are complex eigenvalues with nonzero imaginary parts such that $\lambda_{2l}^h=\overline{\lambda}_{2l-1}^h$ $(m+1\le l\le 2M+1)$. Introduce matrices

$$\begin{aligned} D(\rho) &= \big[\rho^{\lambda_1^h}\xi_1,\ \cdots,\ \rho^{\lambda_{2m}^h}\xi_{2m},\ \mathrm{Re}(\rho^{\lambda_{2m+2}^h}\xi_{2m+2}),\ \mathrm{Im}(\rho^{\lambda_{2m+2}^h}\xi_{2m+2}),\\ &\quad \cdots,\mathrm{Re}(\rho^{\lambda_{4M+2}^h}\xi_{4M+2}),\ \mathrm{Im}(\rho^{\lambda_{4M+2}^h}\xi_{4M+2})\big],\\ E(\rho) &= \big[\rho^{\lambda_1^h-1}\eta_1,\ \cdots,\ \rho^{\lambda_{2m}^h-1}\eta_{2m},\ \mathrm{Re}(\rho^{\lambda_{2m+2}^h-1}\eta_{2m+2}),\ \mathrm{Im}(\rho^{\lambda_{2m+2}^h-1}\eta_{2m+2}),\\ &\quad \cdots,\mathrm{Re}(\rho^{\lambda_{4M+2}^h-1}\eta_{4M+2}),\ \mathrm{Im}(\rho^{\lambda_{4M+2}^h-1}\eta_{4M+2})\big]. \end{aligned}$$

Finally, we get the semi-discrete approximate solution of problem (12):

$$u^h(\rho,\phi) = \Psi(\phi)^T D(\rho)D(1)^{-1}F, \tag{13}$$

$$p^h(\rho,\phi) = \Phi(\phi)^T E(\rho)D(1)^{-1}F. \tag{14}$$

Remark: *We can deal with the Neumann boundary value problem based on the expression of the semi-discrete solution of the Dirichlet BVP given in (13)–(14). In addition, We can similarly define the stress intensity factors (SIFs) K_I and K_{II} at the crack-tip in the incompressible materials.*

Numerical examples

In order to demonstrate the effectiveness of the direct method of lines given in this paper, two numerical examples are discussed. First, we consider the following problem with a corner.

Example 1. We consider the problem

$$-\mu^i\triangle u^i + \mathrm{grad}\,p^i = 0, \qquad \text{in } \Omega_i,\quad 1\le i\le J, \tag{15}$$

$$\mathrm{div}\,u^i = 0, \qquad \text{in } \Omega_i,\quad 1\le i\le J, \tag{16}$$

$$u^i\big|_{\Gamma_i} = f^{\,i}, \qquad 1\le i\le J, \tag{17}$$

$$u^{i-1}\big|_{\theta=\theta_i^-} = u^i\big|_{\theta=\theta_i^+}, \qquad \sigma_n^{i-1}\big|_{\theta=\theta_i^-} = \sigma_n^i\big|_{\theta=\theta_i^+}, \quad 2\le i\le J, \tag{18}$$

$$\sigma_n^1\big|_{\theta=\theta_1} = 0, \qquad \sigma_n^J\big|_{\theta=\theta_{J+1}} = 0, \tag{19}$$

where $J=4$ and Ω_i $(i=1,2,\cdots,4)$ is given in Fig. (3),

$$\begin{aligned} \Gamma_D &= \partial\Omega \setminus \left\{x\in R^2 \,\middle|\, -1\le x_1\le 0,\ x_2=0^- \text{ or } 0\le x_2\le 1,\ x_1=0^+\right\},\\ f &= \begin{pmatrix} f_1\\ f_2\end{pmatrix} = \begin{pmatrix} x_2\\ -x_1\end{pmatrix} \end{aligned}$$

We assume that $\mu^i=2^{i-1}\mu$, for $1\le i\le J$. In all examples we let $\mu=300$. We know the exact solution is $u=(x_2,\,-x_1)^T,\ p=0$.

Let M be an even positive integer, the partition of $[-\pi, \pi/2]$ is given by (10) with $\theta_J = \pi/2$, $h = \frac{3\pi}{4M}$,

$$\begin{aligned} \phi_j &= -\pi + 2(j-1)h, \quad j = 1, 2, \cdots, M+1, \\ \phi_{j+1/2} &= \phi_j + h, \quad\quad\quad j = 1, 2, \cdots, M. \end{aligned} \tag{20}$$

Denoting the numerical solution of (15)–(19) by (u^h, p^h), the results of the first three eigenvalues for the one material case of Example 1 are given in Table 1 for different M, which have been compared with the exact results.

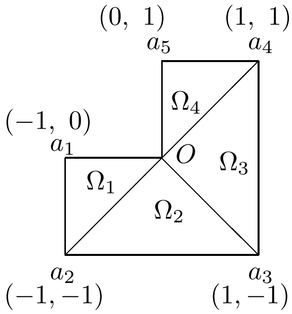

Fig.(3)

Table 1: The results of Example 1.

M	λ_3^h	λ_4^h	λ_5^h	Error
12	0.550231	0.939503	0.997947	2.9338e-2
24	0.549332	0.937753	0.999883	4.3508e-3
48	0.548838	0.937581	0.999986	1.1345e-3

where Error $= \|u - u^h\|_{1,\Omega} + \|p - p^h\|_{0,\Omega}$.

Example 2. We now consider a interface crack problem with Neumann boundary condition (see Fig. (4)):

$$-\mu^i \triangle u^i + \operatorname{grad} p^i = 0, \quad \text{in } \Omega_i, \quad 1 \le i \le J, \tag{21}$$

$$\operatorname{div} u^i = 0, \quad \text{in } \Omega_i, \quad 1 \le i \le J, \tag{22}$$

$$\sigma_n^i\big|_{\Gamma_i} = g^i, \quad 1 \le i \le J, \tag{23}$$

$$u^{i-1}\big|_{\theta=\theta_i^-} = u^i\big|_{\theta=\theta_i^+}, \quad \sigma_n^{i-1}\big|_{\theta=\theta_i^-} = \sigma_n^i\big|_{\theta=\theta_i^+}, \quad 2 \le i \le J, \tag{24}$$

$$\sigma_n^1\big|_{\theta=\theta_1} = 0, \quad \sigma_n^J\big|_{\theta=\theta_{J+1}} = 0, \tag{25}$$

where

$$g(x) = \begin{pmatrix} g_1(x) \\ g_2(x) \end{pmatrix}$$

$$g_1(x) = 0, \; g_2(x) = \begin{cases} -1, & x_2 = -1, \\ 1, & x_2 = 1, \\ 0, & \text{otherwise;} \end{cases}$$

$$\mu^i = 2^{i-1}\mu, \quad 1 \le i \le J.$$

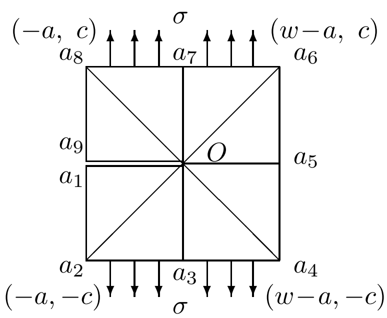

Fig.(4)

Here $J = 8$, $c/w = a/w = 0.5$. Let M be an even positive integer, the partition of $[-\pi, \pi]$ is given by (10) with $\theta_J = \pi$, $h = 2\pi/M$ and

$$\begin{aligned} \phi_j &= -\pi + (j-1)h, \quad j = 1, 2, \cdots, M+1, \\ \phi_{j+1/2} &= \phi_j + h/2, \quad j = 1, 2, \cdots, M. \end{aligned} \tag{26}$$

The results for Example 2 are given in Table 2 for different M, where $K^{h*} = K^h \cdot a^{\lambda_3^h - 1}/\sigma\sqrt{\pi} = K_I^{h*} + iK_{II}^{h*}$. We can see that our method is effective for solving the incompressible problems and calculating SIFs.

Table 2: The results of Example 2.

M	λ_3^h	λ_4^h	λ_5^h	λ_6^h	K_I^{h*}	K_{II}^{h*}
16	0.576618	0.600351	0.964218	0.997352	0.481622	0.0336196
32	0.574633	0.598260	0.956448	0.999875	0.479231	0.0320021
64	0.573838	0.597324	0.953956	0.999993	0.477376	0.0313716

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