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FETI domain decomposition algorithms for sensitivity analysis in contact shape optimization

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Introduction

In this paper, we show that the computational cost of the contact shape optimization may be essentially reduced by the application of a domain decomposition method to the solution of the state variational inequality that describes the equilibrium of a system of elastic bodies. In particular, we describe an algorithm for the minimization of a compliance of one body in a coercive system of bodies during their mutual contacts. After discretization by the finite element method, the algorithm uses a feasible directions method for minimization of the cost functional.

To evaluate gradients of the cost function that are necessary for implementation of the feasible direction method, we describe two different methods for sensitivity analysis. The first one, the so-called overall finite difference method, is based on a simple approximation of partial derivatives of the cost function by the finite differences. It turns out that the decomposition of the stiffness matrices of bodies that have prescribed shape is carried out only once, so that the proposed method of solution of the discretized variational inequality can partly exploit the specific structure of the shape optimization problem. However, the stiffness matrix of the body whose shape is to be designed must be decomposed for each design variable in each design step. This leads naturally to application of the semianalytic method [OL94] that, though algebraically more complicated, works in each design step with

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only one stiffness matrix, so that only one decomposition of one block of the block diagonal stiffness matrix is necessary to carry out one design step, regardless of the number of design variables. The state variational inequality is then solved by an efficient algorithm [ZAS98]

The algorithm has been implemented into the system ODESSY [OL94, RLO93] developed at the Institute of Mechanical Engineering in Aalborg and tested on several model problems. Results of numerical experiments indicate that there are problems for which the algorithm presented is effective.

Discretized contact shape optimization problem

We shall start our exposition from the discretized contact problem. Suppose that $\mathbf{K}$ is the stiffness matrix of order n resulting from the finite element discretization of a system of elastic bodies $\Omega_1, \ldots, \Omega_p$ with enhanced bilateral boundary conditions. With a suitable numbering of nodes, we can achieve that $\mathbf{K} = \mathrm{diag}\,(\mathbf{K}_1, \ldots, \mathbf{K}_p)$, where each $\mathbf{K}_i$ denotes a band matrix which may be identified with the stiffness matrix of the body Ω_i. We assume that $\mathbf{K}$ is positive definite. Let us denote by $\mathbf{f}$ the vector of nodal forces.

Let us now assume that the shape of the first region Ω_1 depends on a vector of design variables α, so that the energy functional will have the form

$$J(\mathbf{u}, \alpha) = \frac{1}{2}\mathbf{u}^T\mathbf{K}(\alpha)\mathbf{u} - \mathbf{f}^T(\alpha)\mathbf{u} \tag{1}$$

where the stiffness matrix $\mathbf{K}(\alpha)$ and possibly the vector of nodal forces $\mathbf{f}(\alpha)$ depend on α. The matrix $\mathbf{N}$ and the vector $\mathbf{c}$ that describe the linearized incremental condition of non-interpenetration also depend on α, so that the solution $\mathbf{u}(\alpha)$ of the contact problem with the region $\Omega_1 = \Omega_1(\alpha)$ satisfies

$$\mathbf{u}(\alpha) = \arg\min J(\mathbf{u}, \alpha) : u \in C(\alpha), \tag{2}$$

where

$$C(\alpha) = \{\mathbf{u} : \mathbf{N}(\alpha)\mathbf{u} \leq \mathbf{c}(\alpha)\}.$$

More details about formulation and discretization of contact problems may be found in Kikuchi and Oden [KO88] or Hlaváček at al [HHNL88].

We shall consider the contact shape optimization problem to find

$$\min\{\mathcal{J}(\alpha) : \alpha \in D_{adm}\} \tag{3}$$

where $\mathcal{J}(\alpha)$ is the cost functional that derives optimality criterion for design of body $\Omega_1(\alpha)$. The set of admissible design variables D_{adm} defines all feasible designs. For example, if the cost functional is defined by $\mathcal{J}(\alpha) \equiv J(\mathbf{u}, \alpha)$, then the minimal compliance problem is obtained. Set of admissible design parameters could be given by

$$D_{adm} = \{l \leq \alpha \leq r : \mathrm{vol}(\Omega(\alpha)) \leq \mathrm{vol}(\Omega(0))\} \tag{4}$$

It has been proved that the minimal compliance problem has at least one solution and that the functional $J(\mathbf{u}, \alpha)$ considered as a function of α has derivatives under natural assumption [HN96].

Duality-based sensitivity analysis

The goal of the sensitivity analysis is to find the influence of design change to the solution of state problem and to the value of the cost function. It means, that we are looking for the directional derivative of solution of the state problem

$$\mathbf{u}'(\alpha,\beta) = \lim_{t\to 0+} \frac{\mathbf{u}(\alpha + t\beta) - \mathbf{u}(\alpha)}{t} \tag{5}$$

where β denotes direction of this directional derivative which is substituted during computation by vectors $\Delta\alpha = (0,\ldots,0,\Delta\alpha_i,0,\ldots,0)^T$ for $i = 1,\ldots,k$, where k is the number of design variables that control the design of bodies.

The simplest method for computation of this derivative is to use the overall forward finite difference approximation $\Delta\mathbf{u}/\Delta\alpha_i$ to the design sensitivity $\partial\mathbf{u}/\partial\alpha_i$ that is given by

$$\frac{\partial\mathbf{u}(\alpha)}{\partial\alpha_i} = \frac{\Delta_i\mathbf{u}(\alpha)}{\Delta\alpha_i} = \frac{\mathbf{u}(\alpha_1,\ldots,\alpha_i+\Delta\alpha_i,\ldots,\alpha_k) - \mathbf{u}(\alpha_1,\ldots,\alpha_k)}{\Delta\alpha_i} \tag{6}$$

It follows that the overall finite difference method for evaluation of the gradient of $\mathbf{u}$ as a function of the design variables α requires $k+1$ solutions of (2). An unpleasant complication is that the Hessian of the quadratic form (1) is different for each auxiliary problem so that we have to carry out $k+1$ times the decomposition of the block K_d that corresponds to the body whose shape is to be computed.

This drawback may be removed by extending the analytic or semianalytic method of sensitivity analysis for problems with state equality [OL94] in contact problems [HN96]. In the rest of this section, the semianalytic aproach will be described.

The Lagrange function of the problem (2) has the form

$$\mathcal{L}(\mathbf{u},\mathbf{x},\alpha) = \frac{1}{2}\mathbf{u}^T\mathbf{K}(\alpha)\mathbf{u} - \mathbf{f}^T(\alpha)\mathbf{u} + \mathbf{x}^T\left(\mathbf{N}(\alpha)\mathbf{u} - \mathbf{c}(\alpha)\right) \tag{7}$$

where $\mathbf{u}$ and $\mathbf{x}$ also depend on the vector of design variables α. For the problem (2) we can prescribe Karush-Kuhn-Tucker conditions in following terms

$$\begin{aligned} &\mathbf{K}(\alpha)\mathbf{u} = \mathbf{f}(\alpha) - \mathbf{N}^T(\alpha)\mathbf{x} \\ &\mathbf{N}(\alpha)\mathbf{u} - \mathbf{c} \le \mathbf{o} \\ &\lambda \ge \mathbf{o} \end{aligned} \tag{8}$$

Let the set $I = \{i : \mathbf{n}_{i*}(\alpha)\mathbf{u} = \mathbf{c}_i(\alpha)\}$ denote set of indices of nodal variables in contact, let $\mathbf{n}_{i*}(\alpha)$ denote the i^{th} row of matrix $\mathbf{N}(\alpha)$ from the problem (2) and let vector $\mathbf{u}$ denote solution of the state problem (2). Further, for analysis of all contact cases we divide the set I to the two sets

$$\begin{aligned} I_s &= \{i : i \in I \wedge \mathbf{x}_i > 0\} \\ I_w &= \{i : i \in I \wedge \mathbf{x}_i = 0\} \end{aligned} \tag{9}$$

where I_s is the set of indices of nodal variables in, so called, strong contact, I_w is the set of indices in weak contact and $\mathbf{x}$ is the solution of the dual formulation of the state problem (2). After formal differentiation of conditions (8) and after some simplification we obtain the new problem

$$\min_{\mathbf{z}\in G(\alpha,\beta)} \mathcal{H}(\alpha,\beta) \tag{10}$$

where

$$\begin{aligned} \mathcal{H}(\alpha,\beta) &= \frac{1}{2}\mathbf{z}^T\mathbf{K}(\alpha)\mathbf{z} - \mathbf{z}^T\left(\mathbf{f}'(\alpha,\beta) - \mathbf{K}'(\alpha,\beta)\mathbf{u} - \mathbf{N}'^T(\alpha,\beta)\mathbf{x}\right) \\ G(\alpha,\beta) &= \big\{\mathbf{z} : \mathbf{n}_{j*}(\alpha)\mathbf{z} \le \mathbf{f}'(\alpha,\beta) - \mathbf{n}'_{j*}(\alpha,\beta)\mathbf{u} \quad \text{for} \quad j \in I_w, \\ &\qquad \mathbf{n}_{j*}(\alpha)\mathbf{z} = \mathbf{f}'(\alpha,\beta) - \mathbf{n}'_{j*}(\alpha,\beta)\mathbf{u} \quad \text{for} \quad j \in I_s\big\} \end{aligned} \tag{11}$$

Symbols $\mathbf{K}'(\alpha,\beta)$, $\mathbf{f}'(\alpha,\beta)$ and $\mathbf{N}'(\alpha,\beta)$ represent directional derivatives in direction β that have the same definition as $\mathbf{u}'(\alpha,\beta)$. At this place it is important to notice that these derivatives can be simply evaluated. It has been proved [HN96] that the solution of this problem is the directional derivative $\mathbf{u}'(\alpha,\beta)$ of solution of problem (2).

Let us make some notifications for simplifying the problem (10)

$$\begin{aligned}
&\tilde{\mathbf{f}}(\alpha,\beta) = \mathbf{f}'(\alpha,\beta) - \mathbf{K}'(\alpha,\beta)\mathbf{u} - \mathbf{N}'^T(\alpha,\beta)\mathbf{x} \\
&\mathbf{N}_w(\alpha) = (\mathbf{n}_{j*}(\alpha))_{j\in I_w}, \qquad \mathbf{c}_w(\alpha,\beta) = \left(\mathbf{f}'(\alpha,\beta) - \mathbf{n}'_{j*}(\alpha,\beta)\mathbf{u}\right)_{j\in I_w} \\
&\mathbf{N}_s(\alpha) = (\mathbf{n}_{j*}(\alpha))_{j\in I_s}, \qquad \mathbf{c}_s(\alpha,\beta) = \left(\mathbf{f}'(\alpha,\beta) - \mathbf{n}'_{j*}(\alpha,\beta)\mathbf{u}\right)_{j\in I_s}
\end{aligned}$$

where $\mathbf{N}_w(\alpha)$,$\mathbf{N}_s(\alpha)$ are matrices that decompose the original matrix $\mathbf{N}(\alpha)$ of contact conditions from problem (2) and $\mathbf{c}_w(\alpha,\beta)$,$\mathbf{c}_s(\alpha,\beta)$ are vectors of dimensions corresponding to number of rows of matrices $\mathbf{N}_w(\alpha)$,$\mathbf{N}_s(\alpha)$. Then, we can rewrite problem (10) in the form

$$\min_{\mathbf{z}\in\tilde{G}(\alpha,\beta)} \tilde{\mathcal{H}}(\alpha,\beta) \tag{12}$$

where

$$\begin{aligned}
\tilde{\mathcal{H}}(\alpha,\beta) &= \frac{1}{2}\mathbf{z}^T\mathbf{K}(\alpha)\mathbf{z} - \tilde{\mathbf{f}}^T(\alpha,\beta)\mathbf{z} \\
\tilde{G}(\alpha,\beta) &= \{\mathbf{z} : \mathbf{N}_w(\alpha)\mathbf{z} \le \mathbf{c}_w(\alpha,\beta), \mathbf{N}_s(\alpha)\mathbf{z} = \mathbf{c}_s(\alpha,\beta)\}
\end{aligned}$$

It is easy to see that the last problem is again a quadratic programming problem with linear constraints in the form of equalities and inequalities. Using the theory of duality, we can convert our problem to the following problem

$$\Phi(\lambda) \to \min \quad \text{subject} \quad \text{to} \quad \lambda \ge \mathbf{o}, \mathbf{M}\lambda = \mathbf{d} \tag{13}$$

where

$$\Phi(\lambda) = \frac{1}{2}\lambda^T\mathbf{N}_w(\alpha)\mathbf{K}^{-1}(\alpha)\mathbf{N}_w^T(\alpha)\lambda - \lambda^T\left(\mathbf{N}_w(\alpha)\mathbf{K}^{-1}(\alpha)\tilde{\mathbf{f}}^T(\alpha,\beta) - \mathbf{c}_w(\alpha,\beta)\right)$$

and

$$\mathbf{M} = \mathbf{N}_s(\alpha)\mathbf{K}^{-1}(\alpha)\mathbf{N}_w^T(\alpha), \mathbf{d} = \mathbf{N}_s(\alpha)\mathbf{K}^{-1}(\alpha)\tilde{\mathbf{f}}^T(\alpha,\beta) - \mathbf{c}_s(\alpha,\beta)$$

Finally, the derivative $\mathbf{u}'(\alpha,\beta)$ can be obtained from equation

$$\mathbf{u}'(\alpha,\beta) = \mathbf{K}^{-1}(\alpha)\left(\tilde{\mathbf{f}}(\alpha,\beta) - \mathbf{N}_w^T(\alpha)\lambda\right).$$

The dual problem (13) with simple inequality constraint and linear equality constraint is efficiently solvable by the algorithm using the augmented Lagrangians with adaptive precision control described by Dostál, Friedlander and Santos in [ZAS96]. Thus the semi-analytic method for sensitivity analysis requires solution of k quadratic programming problems (13) with the same matrix

$$\mathbf{K}^{-1}(\alpha) = \operatorname{diag}\left(\mathbf{K}_1{}^{-1}(\alpha), ..., \mathbf{K}_p{}^{-1}(\alpha)\right).$$

Using this method we exploit not only the advantages of dual formulation of quadratic programming problems, but we can use the decomposition of matrix $\mathbf{K}(\alpha)$ from the solution of the state problem to the sequence of problems in the semi-analytic sensitivity analysis. Thus the semi-analytic approach requires only one decomposition of the stiffness matrix which compares favorably with k+1 decompositions of the overall finite difference approach.

The directional derivatives obtained by the sensitivity analysis can then be exploited for shape optimization. We use sequential linearization in our experiments. More discussion about the outer minimization procedure may be found in Kirsch [Kir94] or Fancello and Feijóo [FF94].

Numerical experiments

We have tested our algorithm on the solution of a simple model problem. The problem was to find the shape of the lower part of the upper body of the system of elastic bodies in Figure 1 so that the compliance of the system is minimal while the volume of the modified upper body does not exceed the volume of the body in the original design. The system has been discretized by the finite element method, so that the discretized system had 1600 nodal variables with possibly 41 nodes in contact. The latter number is the number of dual variables. The Poisson ratio of both bodies was 0.3, the Young modulus of the upper and the lower body was 210000MPa and 100000MPa, respectively. The distributed force with density -1000MPa was acting on the upper surface of the upper body. The bodies were fixed on the right, while zero normal displacements were prescribed on the left and on the bottom of the lower body. It was also required that the bodies do not penetrate in the reference configuration. The design is controlled by vertical movement of six points that are uniformly distributed on the lower boundary of the upper body.

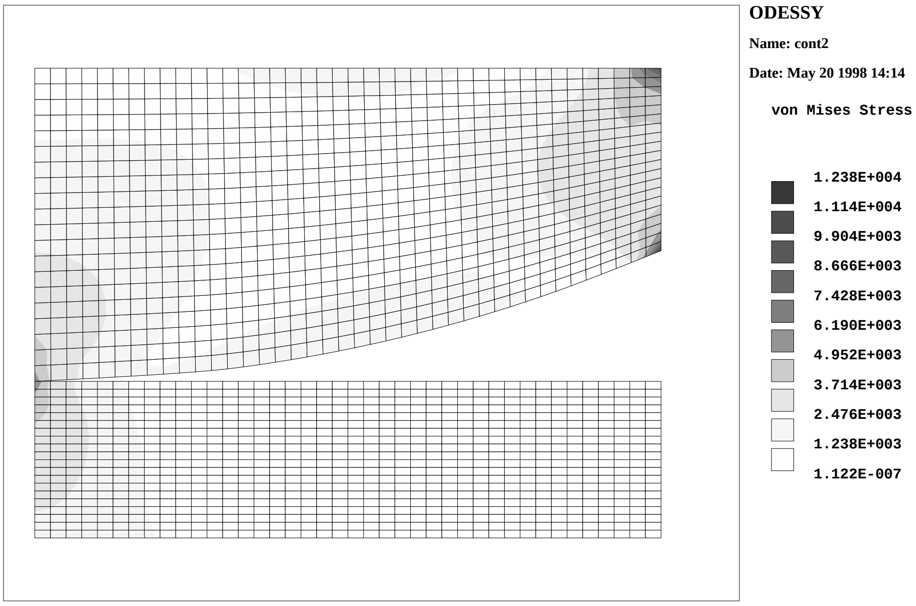

Figure 1 Model problem

The algorithm has been included in the system ODESSY with the overall finite difference sensitivity analysis and sequential linear programming. The performance of the algorithm is given in Table 1 and the final design is depicted in Figure 2. The distribution of the contact pressure is nearly uniform as expected. We explain small variations of the contact stresses by the imposed condition of non-interpenetration in the reference configuration. The computations were carried out on one processor of an IBM SP/2.

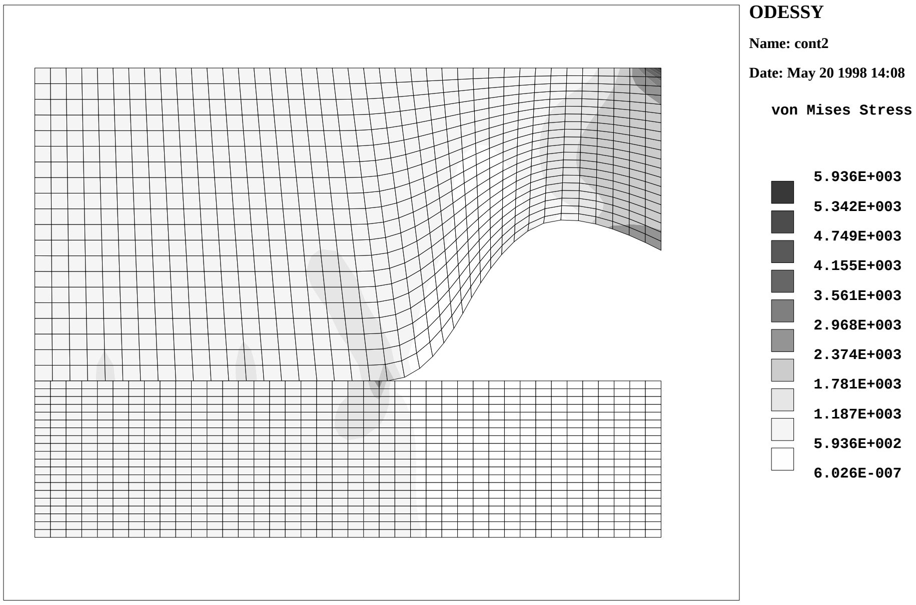

Figure 2 Optimized design

Table 1 Performance in cg iterations and seconds

design	Analysis		Sensitivity anal. Average DVs		Total		Value of cost fun. $\times 10^4$	Change of design %
	iter	time	iter	time	iter	time		
1	13	3	14	4	96	26	1.238	—
2	11	4	11	4	77	30	1.111	4.34
3	13	4	14	5	96	34	0.988	6.21
4	13	5	15	5	101	33	0.823	8.69
5	15	4	17	5	114	32	0.618	8.31
6	16	5	15	5	106	32	0.465	6.03
7	18	6	20	5	135	37	0.367	6.34
8	14	4	14	4	96	27	0.286	5.25
9	16	4	17	4	117	28	0.281	3.47
10	17	5	16	4	114	29	0.278	0.03
11	13	4	15	4	102	28	0.277	0.00
Total	159	48	166	48	1154	336		

Comments and conclusions

A new duality-based method of solution of strictly convex quadratic programming problems has been applied to the minimization of the compliance of a system of elastic bodies. Theoretical results [ZAS98] guarantee the convergence and the robustness of the method. The method has been applied to minimization of compliance of a system of elastic bodies and the efficiency of the method has been confirmed by results of a numerical experiment. The method may be extended to the solution of problems with friction [DV97b, DV97a] and to the solution of semicoercive problems [ZAS96, ZAS98]. The implementation of the algorithm to the solution of these problems is in progress together with implementation of the semianalytic method of sensitivity analysis and a more sophisticated outer minimization procedure. The salient feature of the algorithms presented is essentially the reduction in the cost of decomposition in preparing domain decomposition based solutions for the state variational inequality that is enabled by the special structure of the problem considered. The algorithm may be more efficient in a parallel environment.

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