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A FETI Solver for Corotational Nonlinear Problems

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Introduction

Corotational Finite Element Methods (CFEM) are a way of approaching structural problems where geometric nonlinearities are involved due to large rotations and displacements but small deformations. Such problems are of particular importance to the aerospace industry where structures may exhibit large rotations and flexions, but all components remain within the realm of small deformations. CFEM is nonlinear by design and solution of a single problem requires solving many linear systems with an evolving tangent stiffness matrix. Large systems lead naturally to using the Finite Element Tearing and Interconnecting (FETI) method to solve them. The FETI method is a robust and efficient domain decomposition method for linear structural problems and its use for the resolution of some other classes of nonlinear problems has already been examined by several investigators [Rou95, RR98].

Specific issues to CFEM are related to an unsymmetric tangent stiffness matrix and associated null space which evolves during the nonlinear analysis. We have approached this problem by symmetrizing the tangent stiffness matrix as it was shown to not destroy convergence [Hau94]. Investigations into efficient preconditioning strategies have dealt with reuse of previous Krylov spaces and freezing the preconditioner from one outer iteration to another. Finally, we present a large scale wing type structure undergoing large rotations and flexions.

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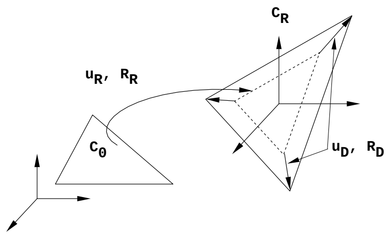

Figure 1 Base and corotated configurations

The Corotational Formulation

The corotational formulation (CR) of geometrical nonlinear structural problems separates rigid body motions from purely deformational motions. Conceptually, in a finite element framework, CR decomposes total displacements into pure rigid body motions and deformational motions at each configuration of nonlinear structural analysis. This can be expressed in vector form for translation degrees of freedom (dofs) and in rotation matrix form for rotation dofs

$$v = v_D + v_R \qquad R = R_D R_R \tag{1}$$

where v_D is deformational motion, v_R is rigid body motion, R_D is the deformational rotation matrix, and R_R is the rigid body rotation matrix. The non-additive behavior of rotation dofs when the rotation axis changes and other geometrical nonlinearities associated with large displacements leads to the nonlinear equilibrium equation

$$F_{int}(x) + F_{ext} = 0 \tag{2}$$

where x contains both v and R, $F_{int}(x)$ is the internal force vector which is dependent on the state of the structure x, and F_{ext} is the external force vector. Eq. (1) are written on an element by element basis. For each element, a reference configuration C_0 is created, and the v_R, R_R displacements express the passage from this configuration to a *corotated* configuration C_R (also called *shadow element*) [dV76]. The element corotated configuration is calculated as a rigid body motion of the element base configuration C_0. From this shadow configuration, the deformational displacements v_d, R_d are used to evaluate the elemental contribution to the internal force vector. As these deformations are small, the usual linear elemental matrices can be used to compute the forces in a frame attached to C_R. After obtaining these local forces, they are transformed back to the global frame (to which C_0 is attached) before summation (see Figure 1). Benefits of using CR over other nonlinear structural analysis description such as the Total Lagrangian method are

- Effective for large rotation/small-strain problems
- Re-use of existing small-strain finite element libraries
- Ability to decouple material nonlinearities from geometric nonlinearities

Maneuvering aircraft undergo large rotations while experiencing small deformational motions. Structural material properties remain linear while nonlinearities are geometric in nature. Adopting CR to model geometrical nonlinear behavior of maneuvering aircraft offers the extra advantage of correctly modeling control surfaces.

Overview of FETI

To keep this paper self-contained, we begin with an overview of the original FETI method [FR94, Far91, FR92]. The general problem to be solved is

$$Ku = F \tag{3}$$

where K is an an $n \times n$ symmetric positive semi-definite sparse matrix arising from the finite element discretization of a second- or fourth-order elastostatic (or elastodynamic) problem defined over a domain Ω, and F is a right hand side n-long vector representing generalized forces. Let Ω be partitioned into a set of N_s *disconnected* subdomains $\Omega^{(s)}$, then the FETI method replaces Eq. (3) with the following equivalent system of subdomain equations

$$\begin{aligned} K^{(s)}u^{(s)} &= F^{(s)} - B^{(s)^T}\lambda \qquad s = 1, \ldots, N_s \\ \Delta &= \sum_{s=1}^{N_s} B^{(s)}u^{(s)} = 0 \end{aligned} \tag{4}$$

where $K^{(s)}$ and $F^{(s)}$ are the unassembled restrictions of K and F to subdomain $\Omega^{(s)}$, λ is a vector of Lagrange multipliers introduced to enforce the constraint $\Delta = 0$ on the subdomain interface boundary $\Gamma_I^{(s)}$, $u^{(s)}$ is the local solution vector, and $B^{(s)}$ is a signed Boolean matrix that describes the subdomain interconnectivity. A detailed derivation of (4) can be found in [FR94, FMR94]. An arbitrary mesh partition may contain $N_f \le N_s$ floating subdomains — that is, subdomains lacking the necessary number of essential boundary conditions needed to prevent the subdomain matrices $K^{(s)}$ from being singular. Therefore N_f of the local Neumann problems

$$K^{(s)}u^{(s)} = F^{(s)} - B^{(s)^T}\lambda \qquad s = 1, \ldots, N_f \tag{5}$$

are ill-posed. Solvability is guaranteed based on the following condition

$$R^{(s)^T}(F^{(s)} - B^{(s)^T}\lambda) = 0 \qquad s = 1, \ldots, N_f \tag{6}$$

where $R^{(s)}$ is the null space of $K^{(s)}$. The solution of Eq. (5) is then given by

$$u^{(s)} = K^{(s)^+}(F^{(s)} - B^{(s)^T}\lambda) + R^{(s)}\alpha^{(s)} \tag{7}$$

where $K^{(s)^+}$ is a generalized inverse of $K^{(s)}$ that need not be explicitly computed [FR92] and $\alpha^{(s)}$ is a vector of six or fewer constants. The extra unknowns $\alpha^{(s)}$ are compensated by additional equations resulting from Eqs. (6) Substituting Eq. (7) into Eq. (4) and using Eq. (6) leads to the FETI interface problem

$$\begin{bmatrix} F_I & -G_I \\ -G_I{}^T & 0 \end{bmatrix} \begin{bmatrix} \lambda \\ \alpha \end{bmatrix} = \begin{bmatrix} d \\ -e \end{bmatrix} \tag{8}$$

where

$$
\begin{aligned}
F_I &= \sum_{s=1}^{N_s} B^{(s)} K^{(s)+} B^{(s)^T}; \qquad G_I = \left[\; B^{(1)} R^{(1)} \quad \ldots \quad B^{(N_f)} B^{(N_f)} \; \right]; \\
\alpha^T &= \left[\; \alpha^{(1)^T} \quad \ldots \quad \alpha^{(N_f)^T} \; \right]; \qquad d = \sum_{s=1}^{N_s} B^{(s)} K^{(s)+} F^{(s)}; \\
e^T &= \left[\; F^{(1)^T} B^{(1)} \quad \ldots \quad F^{(N_f)^T} B^{(N_f)} \; \right]
\end{aligned}
\tag{9}
$$

$$
\begin{aligned}
K^{(s)+} &= K^{(s)^{-1}} \quad \text{if } \Omega^{(s)} \text{ is not a floating subdomain} \\
K^{(s)+} &= \text{a generalized inverse of } K^{(s)}, \text{ if } \Omega^{(s)} \text{ is a floating subdomain}
\end{aligned}
\tag{10}
$$

For structural mechanics and structural dynamics problems, F_I is symmetric since the subdomain stiffness matrices $K^{(s)}$ are symmetric. The objective is to solve by a Preconditioned Conjugate Gradient (PCG) algorithm the interface problem (8) instead of the original problem (3). The PCG algorithm is modified by a projection that enforces iterates λ^k satisfy Eq. (6). The projector P is defined as

$$
P = I - G_I (G_I^T G_I)^{-1} G_I^T \tag{11}
$$

and the FETI algorithm can be written as

$$
\begin{array}{rll}
\text{1. Initialize} & & \\
\lambda^0 &=& G_I \; (G_I{}^T G_I)^{-1} e \\
r^0 &=& d - F_I \lambda^0 \\
\text{2. Iterate } k = 1, \; 2, \; \ldots \text{ until convergence} & & \\
w^{k-1} &=& P^T \; r^{k-1} \\
z^{k-1} &=& \overline{F_I^{-1}} \bar{w}^{k-1} \\
y^{k-1} &=& P \; \bar{z}^{k-1} \\
\zeta^k &=& y^{k-1^T} w^{k-1} / y^{k-2^T} w^{k-2} \quad (\zeta^1 = 0) \\
p^k &=& y^{k-1} + \zeta^k p^{k-1} \quad (p^1 = y^0) \\
\nu^k &=& y^{k-1^T} w^{k-1} / p^{k^T} F_I p^k \\
\lambda^k &=& \lambda^{k-1} + \nu^k p^k \\
r^k &=& r^{k-1} - \nu^k F_I p^k
\end{array}
\tag{12}
$$

The reader can check that the FETI algorithm results in applying PCG to:

$$
A\lambda = d \tag{13}
$$

where $A = P^T F_I P$.

Use of FETI for Corotational Problems

We wish to solve Eq. (4) using a Newton-Raphson approach which leads us to solving the following set of successive linear systems

$$
\begin{aligned}
K_T(x_0) \Delta u_1 &= f_1 \\
K_T(x_{i-1}) \Delta u_i &= f_i \\
K_T(x_{n-1}) \Delta u_n &= f_n
\end{aligned}
\tag{14}
$$

where $K_T(x_{i-1})$ is the i^{th} configurations tangent stiffness matrix, x_{i-1} is the previous structural state, Δu_i is the i^{th} incremental displacement vector, f_i is the i^{th} right hand side vector which depends on the solution strategy [Rik72], and n is the number of iterations for convergence. After each linear solve, x_i is updated using the incremental displacement. Convergence of Newton-Raphson is based on reaching these criteria

$$||\Delta u||_2 < tolerance_{disp} \qquad ||f||_2 < tolerance_{residual} \tag{15}$$

The tangent stiffness matrices are generally unsymmetric before convergence is reached. In this work we used a symmetrized version of the tangent matrices, as it is required by the FETI method and has been shown not to harm convergence [Hau94].

Solution of Nearby Linear Systems

Multiple Left Hand Sides (MLHS)

We will refer to the method described in the previous paragraph as the Full Newton (FN) method. The tangent stiffness matrix is rebuilt at each nonlinear iteration, resulting in a robust geometrical nonlinear structural solution algorithm.

The use of FETI to solve Eqs. (14) creates a set of systems similar to Eq. (13). Solving each system by PCG can be viewed as a minimization problem of the form:

$$\min_{\lambda}(\Phi_i) = 1/2\lambda^T A_i \lambda - d_i^T \lambda \qquad i = 1, \ldots, n \tag{16}$$

To solve each system, a Krylov space is generated by the PCG algorithm. This space is spanned by the set of search directions. Let us denote the set of search directions for the linear system i by

$$S_i = \{s_i^1, s_i^2, \ldots, s_i^k, \ldots, s_i^{r_i}\} \qquad i = 1, \ldots, n \tag{17}$$

The search directions computed within a Newton iteration are orthogonal with respect to A_i. However, since the tangent stiffness matrix is changing over successive Newton iterations, the search directions are not orthogonal across Newton iterations. Nonetheless, the successive A_i matrices can be expected to be spectrally close one to another. This observation suggests that at step i we can make use of the previous sets S_i to define a preconditioner [Rou95]. The approach is to replace the usual preconditioned residual vector Pr by a modified vector $\hat{Pr}$ given by:

$$\hat{Pr} = Pr + \sum_{j=1}^{i-1} S_j y_j \tag{18}$$

where the vectors y_j are successively chosen to satisfy the minimization problem

$$\min_{y_j}(\Phi_j) = 1/2\lambda^T A_j \lambda - r^T \lambda \tag{19}$$

with $\lambda = Pr + \sum_{l=1}^{j} S_l y_l$. Note that the function being minimized in this case is similar to that given in Eq. (16) when A_i is replaced with A_j and d_i is replaced by the current

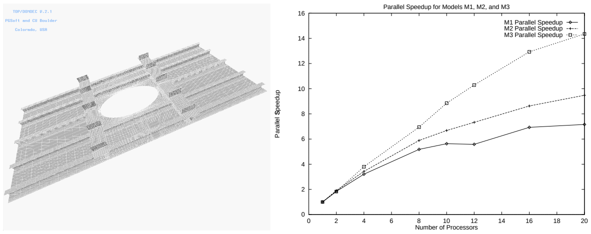

Figure 2 V-22 Wing Panel, Parallel Speedup for M1, M2 and M3

PCG residual r. All computations performed, we obtain the expression for y_j:

$$y_j = (S_j^T A_j S_j)^{-1}(S_j^T r - S_j^T A_j(Pr + \sum_{l=1}^{j-1} S_l y_l)) \qquad j = 1, \ldots, i-1 \tag{20}$$

The assumption is that successive tangent stiffness matrices are *similar* in some sense and thus the preconditioner will be of benefit. Due to the corotational formulation, the null space of the subdomain tangent matrices $K_T^{(s)}$ evolves in time. This implies that the projector must be correctly recomputed at each iteration and the space of search directions must contain projected vectors.

Multiple Right Hand Sides (MRHS)

An alternative approach to solve Eq. (14) is to use a Newton Like (NL) Method where the tangent stiffness matrix may be "frozen" and reused for subsequent nonlinear iterations. In the extreme case, the initial tangent stiffness matrix may be used for the entire simulation but then convergence of the outer Newton iterations is not guaranteed. The effect of using a NL method versus a FN method is to slow down convergence and possibly increase the number of linear solutions.

In contrast to the FN method, we now solve with a fixed A_f matrix for a set of varying right hand sides. Therefore, as has been done for linear dynamic problems [Far95], the set of search directions S_i can be kept A_f orthogonal to all previous sets. The initial λ_i^0 must now be modified to include the contribution of all previous sets and then orthogonality of the search directions with respect to A_f enforced at each FETI iteration.

We can of course combine the MRHS approach with the Krylov Space preconditioning of MLHS. In this case, the matrix A_f is changed every m iteration, and we can apply the preconditioning of Eqs. (18) in which case the sets S_i are the union of all sets generated by the same tangent matrix A_f.

Numerical Example: Composite Wing Panel

The composite wing panel from a V-22 tilt-rotor aircraft [Dav91] considered here contains design features such as ply drop-offs, ply interleaves, axial stiffeners, transverse ribs, clips, brackets and a large elliptical access hole. The panel is clamped at one end and loaded on the opposite end by prescribed displacements. The finite element models, designated M1, M2, and M3 have 56916, 223620, and 885924 d.o.f. respectively (see Figure 2). Each model is composed of 47 composite materials and discretized with three node ANS shell elements.

The results in Table 1 were generated with the 2-Level FETI Method [FM98] for fourth order elastostatic problems on a SGI Onyx 2000 using 16 processors. FETI convergence was monitored using a tolerance of 1.0E-3 while Newton's convergence was checked using 1.0E-5. These values were chosen due to quadratic convergence properties of Newton's method. The optimal Dirichlet FETI preconditioner was selected for this problem. Using a NL method, this problem requires two load steps and eight Newton iterations while FN requires one load step and four iterations to converge. Both methods were tested with and without Krylov acceleration methods to compare iteration counts, CPU, and memory requirements. For FN, K_T, and FETI preconditioner F_D^{-1} are rebuilt at each Newton iteration while for NL method, rebuilding occurs at the start of a load step. The most evident conclusion is the

Table 1 Composite Panel Model, M3 (885924 d.o.f., 250 Subdomains)

Load Step	Newton Itr.	NL-KLR	NL-KR	FN-KL	FN
1	1	143 itr.	143 itr.	143 itr.	143 itr.
	2	72 itr.	72 itr.	171 itr.	198 itr.
	3	36 itr.	36 itr.	108 itr.	198 itr.
	4	19 itr.	19 itr.	46 itr.	198 itr.
2	5	124 itr.	168 itr.	- itr.	- itr.
	6	57 itr.	64 itr.	- itr.	- itr.
	7	36 itr.	33 itr.	- itr.	- itr.
	8	16 itr.	14 itr.	- itr.	- itr.
	Total	503 itr.	549 itr.	**468 itr.**	737 itr.
	FETI CPU	**940 sec.**	1078 sec.	979 sec.	1513 sec.
	FETI Mem.	3710 Mb.	3417 Mb.	3657 Mb.	**3307 Mb.**

approximately 40% decrease in CPU when a Krylov acceleration technique is applied. The other conclusion is a small 10% increase in memory requirements associated with the Krylov acceleration methods over the FN method. This can be attributed to the number of stored search direction vectors for each algorithm. The FN with Krylov left hand side acceleration (KL) had the lowest total iteration count and was roughly the same CPU time as NL with Krylov right hand side acceleration (KR). When both Krylov accelerators (KLR) are applied to the NL method, there is a slight improvement in total iteration count and CPU time. Figure 2 shows almost linear parallel speedup for each model. This trend suggests that larger problems will gain greater speedups with increasing numbers of processors. Optimal mesh decompositions of 40 (M1), 140 (M2), and 250 (M3) subdomains were used to compute these speedups [LP98].

Conclusion

We have shown the FETI method in conjunction with CFEM concept to be efficient in solving large scale geometrical nonlinear problems undergoing large rotations and small deformations. Future research will involve implementing the FETI solver and CFEM in a nonlinear dynamic algorithm to solve time-dependent geometrical nonlinear problems such as maneuvering aircraft.

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